

# Extremal Functions for Singular Moser-Trudinger Embeddings

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## Abstract

We study Moser-Trudinger type functionals in the presence of singular potentials. In particular we propose a proof of a singular Carleson-Chang type estimate by means of Onofri's inequality for the unit disk in  $\mathbb{R}^2$ . Moreover we extend the analysis of [1] and [8] considering Adimurthi-Druet type functionals on compact surfaces with conical singularities and discussing the existence of extremals for such functionals.

## 1 Introduction

Let  $\Omega \subseteq \mathbb{R}^2$  be a bounded domain, from the well known Sobolev's inequality

$$\|u\|_{L^{\frac{2p}{2-p}}(\Omega)} \leq S_p \|\nabla u\|_{L^p(\Omega)} \quad p \in (1, 2), \quad u \in W_0^{1,p}(\Omega), \quad (1)$$

one can deduce that the Sobolev space  $H_0^1(\Omega) := W_0^{1,2}(\Omega)$  is embedded into  $L^q(\Omega) \forall q \geq 1$ . A much more precise result was proved in 1967 by Trudinger [27]: on bounded subsets of  $H_0^1(\Omega)$  one has uniform exponential-type integrability. Specifically, there exists  $\beta > 0$  such that

$$\sup_{u \in H_0^1(\Omega), \int_{\Omega} |\nabla u|^2 dx \leq 1} \int_{\Omega} e^{\beta u^2} dx < +\infty. \quad (2)$$

This inequality was later improved by Moser in [20], who proved that the sharp exponent in (2) is  $\beta = 4\pi$ , that is

$$\sup_{u \in H_0^1(\Omega), \int_{\Omega} |\nabla u|^2 dx \leq 1} \int_{\Omega} e^{4\pi u^2} dx < +\infty, \quad (3)$$

and

$$\sup_{u \in H_0^1(\Omega), \int_{\Omega} |\nabla u|^2 dx \leq 1} \int_{\Omega} e^{\beta u^2} dx = +\infty \quad (4)$$

for  $\beta > 4\pi$ . An interesting question consists in studying the existence of extremal functions for (3). Indeed, while there is no function realizing equality in (1), one can prove that the supremum in (3) is always attained. This was proved in [4] by Carleson and Chang for the unit disk  $D \subseteq \mathbb{R}^2$ , and by Flucher ([9]) for arbitrary bounded domains (see also [23] and [15]). The

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proof of these results is based on a concentration-compactness alternative stated by P. L. Lions ([16]): for a sequence  $u_n \in H_0^1(\Omega)$  such that  $\|\nabla u_n\|_{L^2(\Omega)} = 1$  one has, up to subsequences, either

$$\int_{\Omega} e^{4\pi u_n^2} dx \rightarrow \int_{\Omega} e^{4\pi u^2} dx$$

where  $u$  is the weak limit of  $u_n$ , or  $u_n$  concentrates in a point  $x \in \overline{\Omega}$ , that is

$$|\nabla u|^2 dx \rightharpoonup \delta_x \quad \text{and} \quad u_n \rightharpoonup 0. \quad (5)$$

The key step in [4] consists in proving that if a sequence of radially symmetric functions  $u_n \in H_0^1(D)$  concentrates at 0, then

$$\limsup_{n \rightarrow \infty} \int_D e^{4\pi u_n^2} dx \leq \pi(1 + e). \quad (6)$$

Since for the unit disk the supremum in (3) is strictly greater than  $\pi(1 + e)$ , one can exclude concentration for maximizing sequences by means of (6) and therefore prove existence of extremal functions for (3). In [9] Flucher observed that concentration at arbitrary points of a general domain  $\Omega$  can always be reduced, through properly defined rearrangements, to concentration of radially symmetric functions on the unit disk. In particular he proved that if  $u_n \in H_0^1(\Omega)$  satisfies  $\|\nabla u_n\|_2 = 1$  and (5), then

$$\limsup_{n \rightarrow \infty} \int_{\Omega} e^{4\pi u_n^2} dx \leq \pi e^{1+4\pi A_{\Omega}(x)} + |\Omega|. \quad (7)$$

where  $A_{\Omega}(x)$  is the Robin function of  $\Omega$ , that is the trace of the regular part of the Green function of  $\Omega$ . He also proved

$$\sup_{u \in H_0^1(\Omega), \int_{\Omega} |\nabla u|^2 dx \leq 1} \int_{\Omega} e^{4\pi u^2} dx > \pi e^{1+4\pi \max_{\overline{\Omega}} A_{\Omega}} + |\Omega|, \quad (8)$$

which implies the existence of extremals for (3) on  $\Omega$ . Similar results hold if  $\Omega$  is replaced by a smooth closed surface  $(\Sigma, g)$ . Let us denote

$$\mathcal{H} := \left\{ u \in H^1(\Sigma) : \int_{\Sigma} |\nabla u|^2 dv_g \leq 1, \int_{\Sigma} u dv_g = 0 \right\}.$$

Fontana [10] proved that

$$\sup_{u \in \mathcal{H}} \int_{\Sigma} e^{4\pi u^2} dv_g < +\infty \quad (9)$$

and

$$\sup_{u \in \mathcal{H}} \int_{\Sigma} e^{\beta u^2} dv_g = +\infty \quad (10)$$

$\forall \beta > 4\pi$ . Existence of extremal functions for (9) was proved in [13] (see also [12], [14]), again by excluding concentration for maximizing sequences.

In this paper we are interested in Moser-Trudinger type inequalities in the presence of singular potentials. The simplest example is given by the singular metric  $|x|^{2\alpha}|dx|^2$  on a bounded domain  $\Omega \subset \mathbb{R}^2$  containing the origin. In [2] Adimurthi and Sandeep observed that  $\forall \alpha \in (-1, 0]$ ,

$$\sup_{u \in H_0^1(\Omega), \int_{\Omega} |\nabla u|^2 dx \leq 1} \int_{\Omega} |x|^{2\alpha} e^{4\pi(1+\alpha)u^2} dx < +\infty, \quad (11)$$

and

$$\sup_{u \in H_0^1(\Omega), \int_{\Omega} |\nabla u|^2 dx \leq 1} \int_{\Omega} |x|^{2\alpha} e^{\beta u^2} dx = +\infty, \quad (12)$$

for any  $\beta > 4\pi(1 + \alpha)$ . Existence of extremals for (11) has recently been proved in [7] and [8]. The strategy is similar to the one used for the case  $\alpha = 0$ . One can exclude concentration for maximizing sequences using the following estimate, which can be obtained from (6) using a simple change of variables (see [2], [8]).

**Theorem 1.1.** *Let  $u_n \in H_0^1(D)$  be such that  $\int_D |\nabla u_n|^2 dx \leq 1$  and  $u_n \rightharpoonup 0$  in  $H_0^1(D)$ , then  $\forall \alpha \in (-1, 0]$  we have*

$$\limsup_{n \rightarrow \infty} \int_D |x|^{2\alpha} e^{4\pi(1+\alpha)u_n^2} dx \leq \frac{\pi(1+e)}{1+\alpha}. \quad (13)$$

In the first part of this work we will give a simplified version of the argument in [4] and show that (6) (and therefore (13)) can be deduced from Onofri's inequality for the unit disk.

**Proposition 1.1** (See [21], [3]). *For any  $u \in H_0^1(D)$  we have*

$$\log \left( \frac{1}{\pi} \int_D e^u dx \right) \leq \frac{1}{16\pi} \int_D |\nabla u|^2 dx + 1. \quad (14)$$

Theorem 1.1 can be used to prove existence of extremals for several generalized versions of (3). In [1] Adimurthi and Druet proved that

$$\sup_{u \in H_0^1(\Omega), \int_{\Omega} |\nabla u|^2 dx \leq 1} \int_{\Omega} e^{4\pi u^2(1+\lambda\|u\|_{L^2(\Omega)}^2)} dx < +\infty \quad (15)$$

for any  $\lambda < \lambda(\Omega)$ , where  $\lambda(\Omega)$  is the first eigenvalue of  $-\Delta$  with respect to Dirichlet boundary conditions. This bound on  $\lambda$  is sharp, that is

$$\sup_{u \in H_0^1(\Omega), \int_{\Omega} |\nabla u|^2 dx \leq 1} \int_{\Omega} e^{4\pi u^2(1+\lambda(\Omega)\|u\|_{L^2(\Omega)}^2)} dx = \infty. \quad (16)$$

Existence of extremal functions for sufficiently small  $\lambda$  for this improved inequality has been proved in [17] and [28]. Similar results hold for compact surfaces on the space  $\mathcal{H}$ . We refer to [25], [29] and references therein for further improved inequalities.

In this work we will focus on Adimurthi-Druet type inequalities on compact surfaces with conical singularities. Given a smooth, closed Riemannian surface  $(\Sigma, g)$ , and a finite number of points  $p_1, \dots, p_m \in \Sigma$  we will consider functionals of the form

$$E_{\Sigma, h}^{\beta, \lambda, q}(u) := \int_{\Sigma} h e^{\beta u^2(1+\lambda\|u\|_{L^q(\Sigma, g)}^2)} dv_g \quad (17)$$

where  $\lambda, \beta \geq 0$ ,  $q > 1$  and  $h \in C^1(\Sigma \setminus \{p_1, \dots, p_m\})$  is a positive function satisfying

$$h(x) \approx d(x, p_i)^{2\alpha_i} \quad \text{with} \quad \alpha_i > -1 \quad \text{near} \quad p_i \quad i = 1, \dots, m. \quad (18)$$

One of the main motivations for the choice of these singular weights comes indeed from the study of surfaces with conical singularities. We recall that a smooth metric  $\bar{g}$  on  $\Sigma \setminus \{p_1, \dots, p_m\}$  is said

to have conical singularities of order  $\alpha_1, \dots, \alpha_m$  in  $p_1, \dots, p_m$  if  $\bar{g} = hg$  with  $g$  smooth metric on  $\Sigma$  and  $0 < h \in C^\infty(\Sigma \setminus \{p_1, \dots, p_m\})$  satisfying (18). Thus the functional (17) naturally appears in the analysis of Moser-Trudinger embeddings for the singular surface  $(\Sigma, \bar{g})$  (see [26]).

If  $m = 0$  and  $h \equiv 1$ ,  $E_{\Sigma,1}^{\beta,\lambda,q}$  corresponds to the functional studied in [17]. In particular, one has

$$\sup_{u \in \mathcal{H}} E_{\Sigma,1}^{4\pi,\lambda,q} < +\infty \iff \lambda < \lambda_q(\Sigma, g), \quad (19)$$

where

$$\lambda_q(\Sigma, g) := \inf_{u \in \mathcal{H}} \frac{\int_{\Sigma} |\nabla u|^2 dv_g}{\|u\|_{L^q(\Sigma, g)}^2}.$$

As it happens for (11), if  $h$  has singularities the critical exponent becomes smaller. More precisely, in [26] Troyanov (see also [5]) proved that if  $h$  is a positive function satisfying (18), then

$$\sup_{u \in \mathcal{H}} E_{\Sigma,h}^{\beta,0,q} < +\infty \iff \beta \leq 4\pi(1 + \bar{\alpha}) \quad (20)$$

where  $\bar{\alpha} = \min \left\{ 0, \min_{1 \leq i \leq m} \alpha_i \right\}$ . In this paper we combine (19) and (20) obtaining the following singular version of (19).

**Theorem 1.2.** *Let  $(\Sigma, g)$  be a smooth, closed, surface. If  $h \in C^1(\Sigma \setminus \{p_1, \dots, p_m\})$  is a positive function satisfying (18), then  $\forall \beta \in [0, 4\pi(1 + \bar{\alpha})]$  and  $\lambda \in [0, \lambda_q(\Sigma, g))$  we have*

$$\sup_{u \in \mathcal{H}} E_{\Sigma,h}^{\beta,\lambda,q}(u) < +\infty, \quad (21)$$

and the supremum is attained if  $\beta < 4\pi(1 + \bar{\alpha})$  or if  $\beta = 4\pi(1 + \bar{\alpha})$  and  $\lambda$  is sufficiently small. Moreover

$$\sup_{u \in \mathcal{H}} E_{\Sigma,h}^{\beta,\lambda,q}(u) = +\infty$$

for  $\beta > 4\pi(1 + \bar{\alpha})$ , or  $\beta = 4\pi(1 + \bar{\alpha})$  and  $\lambda > \lambda_q(\Sigma, g)$ .

Note that we do not treat the case  $\beta = 4\pi(1 + \bar{\alpha})$  and  $\lambda = \lambda_q(\Sigma, g)$  (see Remark 5.1). In Theorem 1.2, it is possible to replace  $\|\cdot\|_{L^q(\Sigma, g)}$ ,  $\lambda_q(\Sigma, g)$  and  $\mathcal{H}$ , with  $\|\cdot\|_{L^q(\Sigma, g_h)}$ ,  $\lambda_q(\Sigma, g_h)$  and

$$\mathcal{H}_h := \left\{ u \in H_0^1(\Sigma) : \int_{\Sigma} |\nabla_{g_h} u|^2 dv_{g_h} \leq 1, \int_{\Sigma} u dv_{g_h} = 0 \right\},$$

where  $g_h := hg$ . In particular we can extend the Adimurthi-Druet inequality to compact surfaces with conical singularities.

**Theorem 1.3.** *Let  $(\Sigma, g)$  be a closed surface with conical singularities of order  $\alpha_1, \dots, \alpha_m > -1$  in  $p_1, \dots, p_m \in \Sigma$ . Then for any  $0 \leq \lambda < \lambda_q(\Sigma, g)$  we have*

$$\sup_{u \in \mathcal{H}} \int_{\Sigma} e^{4\pi(1+\bar{\alpha})u^2(1+\lambda\|u\|_{L^q(\Sigma, g)}^2)} dv_g < +\infty,$$

and the supremum is attained for  $\beta < 4\pi(1 + \bar{\alpha})$  or for  $\beta = 4\pi(1 + \bar{\alpha})$  and sufficiently small  $\lambda$ . Moreover

$$\sup_{u \in \mathcal{H}} \int_{\Sigma} e^{\beta u^2(1+\lambda\|u\|_{L^q(\Sigma, g)}^2)} dv_g = +\infty,$$

if  $\beta > 4\pi(1 + \bar{\alpha})$  or  $\beta = 4\pi(1 + \bar{\alpha})$  and  $\lambda > \lambda_q(\Sigma, g)$ .

As in [13], [29] and [17], our techniques can be adapted to treat the case of compact surfaces with boundary.

**Theorem 1.4.** *Let  $(\Sigma, g)$  be a smooth compact Riemannian surface with boundary. If  $p_1, \dots, p_m \in \Sigma \setminus \partial\Sigma$  and  $h \in C^1(\Sigma \setminus \{p_1, \dots, p_m\})$  satisfies (18), then  $\forall \beta \in [0, 4\pi(1 + \overline{\alpha})]$  and  $\lambda \in [0, \lambda_q(\Sigma, g))$  we have*

$$\sup_{u \in H_0^1(\Sigma), \int_{\Sigma} |\nabla u|^2 dv_g \leq 1} E_{\Sigma, h}^{\beta, \lambda, q}(u) < +\infty$$

*and the supremum is attained if  $\beta < 4\pi(1 + \overline{\alpha})$  or if  $\beta = 4\pi(1 + \overline{\alpha})$  and  $\lambda$  is sufficiently small. Furthermore if  $\beta > 4\pi(1 + \overline{\alpha})$ , or  $\beta = 4\pi(1 + \overline{\alpha})$  and  $\lambda \geq \lambda_q(\Sigma, g)$ , we have*

$$\sup_{u \in H_0^1(\Sigma), \int_{\Sigma} |\nabla u|^2 dv_g \leq 1} E_{\Sigma, h}^{\beta, \lambda, q}(u) = +\infty.$$

In particular, if  $\Sigma = \overline{\Omega}$  is the closure of a bounded domain in  $\mathbb{R}^2$ , Theorem 1.4 gives the following generalization of the results in [9], [1], [8].

**Corollary 1.1.** *Let  $\Omega \subseteq \mathbb{R}^2$  be a bounded domain. For any choice of  $V \in C^1(\overline{\Omega})$ ,  $V > 0$ ,  $\alpha_1, \dots, \alpha_m > -1$ ,  $x_1, \dots, x_m \in \Omega$ ,  $q > 1$  and  $\lambda \in [0, \lambda_q(\Omega))$ , the supremum*

$$\sup_{u \in H_0^1(\Omega), \int_{\Omega} |\nabla u|^2 dx \leq 1} \int_{\Omega} V(x) \prod_{i=1}^m |x - x_i|^{2\alpha_i} e^{4\pi(1+\overline{\alpha})u^2 (1+\lambda\|u\|_{L^q(\Omega)}^2)} dx$$

*is finite. Moreover it is attained if  $\lambda$  is sufficiently small.*

This paper is organized as follows. Section 2 contains a simple proof of Theorem 1.1. Theorem 1.2 will be proved in the remaining three sections. In section 3 we will state some useful lemmas and prove existence of extremals for  $E_{\Sigma, h}^{\beta, \lambda, q}$  with  $\beta < 4\pi(1 + \overline{\alpha})$ . In Section 4 we will deal with the blow-up analysis for maximizing sequences for the critical case  $\beta = 4\pi(1 + \overline{\alpha})$  and we will prove an estimate similar to (7), which implies the finiteness of the supremum in (21). Finally, in Section 5 we test the functionals with a properly defined family of functions and complete the proof Theorem 1.2. In the Appendix we will discuss some Onofri-type inequalities. In particular we will show how to deduce (14) from the standard Onofri inequality on  $S^2$  and discuss its extensions to singular disks. The proof of Theorems 1.3 and 1.4 is very similar to the one of Theorem 1.2, hence it will not be discussed in this work.

## 2 A Carleson-Chang type estimate.

In this section we will prove Theorem 1.1 by means of (14). We will consider the space

$$H := \left\{ u \in H_0^1(D) : \int_D |\nabla u|^2 dx \leq 1 \right\}$$

and, for any  $\alpha \in (-1, 0]$ , the functional

$$E_{\alpha}(u) := \int_D |x|^{2\alpha} e^{4\pi(1+\alpha)u^2} dx.$$

By (11) we have  $\sup_H E_{\alpha} < +\infty$ . For any  $\delta > 0$ , we will denote with  $D_{\delta}$  the disk with radius  $\delta$  centered at 0.

**Remark 2.1.** *With a trivial change of variables, one immediately gets that if  $\delta > 0$  and  $u \in H_0^1(D_\delta)$  are such that  $\int_{D_\delta} |\nabla u_n|^2 dx \leq 1$ , then*

$$\int_{D_\delta} |x|^{2\alpha} e^{4\pi(1+\alpha)u^2} dx \leq \delta^{2(1+\alpha)} \sup_H E_\alpha.$$

In order to control the values of the Moser-Trudinger functional on a small scale, we will need the following scaled version of (14) (cfr. Lemma 1 in [4]).

**Corollary 2.1.** *For any  $\delta, \tau > 0$  and  $c \in \mathbb{R}$  we have*

$$\int_{D_\delta} e^{cu} dx \leq \pi e^{1+\frac{c^2\tau}{16\pi}} \delta^2$$

*for any  $u \in H_0^1(D_\delta)$  such that  $\int_{D_\delta} |\nabla u|^2 dx \leq \tau$ .*

As in the original proof in [4], we will first assume  $\alpha = 0$  and work with radially symmetric functions. For this reason we introduce the spaces

$$H_{0,rad}^1(D) := \{u \in H_0^1(D) : u \text{ is radially symmetric and decreasing}\}.$$

and

$$H_{rad} := H \cap H_{0,rad}^1(D).$$

Functions in  $H_{rad}$  satisfy the following useful decay estimate.

**Lemma 2.1.** *For any  $u \in H_{rad}$  we have*

$$u(x)^2 \leq -\frac{1}{2\pi} \left( 1 - \int_{D_{|x|}} |\nabla u|^2 dy \right) \log |x| \quad \forall x \in D \setminus \{0\}.$$

*Proof.* We bound

$$\begin{aligned} |u(x)| &\leq \int_{|x|}^1 |u'(t)| dt \leq \left( \int_{|x|}^1 t u'(t)^2 dt \right)^{\frac{1}{2}} (-\log |x|)^{\frac{1}{2}} \\ &\leq \frac{1}{\sqrt{2\pi}} \left( \int_{D \setminus D_{|x|}} |\nabla u|^2 dy \right)^{\frac{1}{2}} (-\log |x|)^{\frac{1}{2}} \\ &\leq \frac{1}{\sqrt{2\pi}} \left( 1 - \int_{D_{|x|}} |\nabla u|^2 dy \right)^{\frac{1}{2}} (-\log |x|)^{\frac{1}{2}}. \end{aligned}$$

□

On a sufficiently small scale, it is possible to control  $E_0$  using only Corollary 2.1 Lemma 2.1 and Remark 2.1.

**Lemma 2.2.** *If  $u_n \in H_{rad}$  and  $\delta_n \rightarrow 0$  satisfy*

$$\int_{D_{\delta_n}} |\nabla u_n|^2 dx \rightarrow 0, \quad (22)$$

*then*

$$\limsup_{n \rightarrow \infty} \int_{D_{\delta_n}} e^{4\pi u_n^2} dx \leq \pi e.$$

*Proof.* Take  $v_n := u_n - u_n(\delta_n) \in H_0^1(D_{\delta_n})$  and set  $\tau_n := \int_{D_{\delta_n}} |\nabla u_n|^2 dx$ . If  $\tau_n = 0$ , then  $u_n \equiv u_n(\delta_n)$  in  $D_{\delta_n}$  and, using Lemma 2.1, we find

$$\int_{D_{\delta_n}} e^{4\pi u_n^2} dx = \pi \delta_n^2 e^{4\pi u_n(\delta_n)^2} \leq \pi < \pi e.$$

Thus, w.l.o.g. we can assume  $\tau_n > 0$  for every  $n \in \mathbb{N}$ . By Holder's inequality and Remark 2.1 we have

$$\begin{aligned} \int_{D_{\delta_n}} e^{4\pi u_n^2} dx &= e^{4\pi u_n(\delta_n)^2} \int_{D_{\delta_n}} e^{4\pi v_n^2 + 8\pi u_n(\delta_n)v_n} dx \\ &\leq e^{4\pi u_n(\delta_n)^2} \left( \int_{D_{\delta_n}} e^{4\pi \frac{v_n^2}{\tau_n}} dx \right)^{\tau_n} \left( \int_{D_{\delta_n}} e^{\frac{8\pi u_n(\delta_n)v_n}{1-\tau_n}} dx \right)^{1-\tau_n} \\ &\leq e^{4\pi u_n(\delta_n)^2} \left( \delta_n^2 \sup_H E_0 \right)^{\tau_n} \left( \int_{D_{\delta_n}} e^{\frac{8\pi u_n(\delta_n)v_n}{1-\tau_n}} dx \right)^{1-\tau_n}. \end{aligned} \quad (23)$$

Applying Corollary 2.1 with  $\tau = \tau_n$ ,  $\delta = \delta_n$  and  $c = \frac{8\pi u_n(\delta_n)}{1-\tau_n}$  we find

$$\int_{D_{\delta_n}} e^{\frac{8\pi u_n(\delta_n)v_n}{1-\tau_n}} dx \leq \delta_n^2 \pi e^{1 + \frac{4\pi u_n(\delta_n)^2}{(1-\tau_n)^2} \tau_n}$$

thus from (23) it follows

$$\begin{aligned} \int_{D_{\delta_n}} e^{4\pi u_n^2} dx &\leq \delta_n^2 \left( \sup_H E_0 \right)^{\tau_n} (\pi e)^{1-\tau_n} e^{4\pi u_n^2(\delta_n) + \frac{4\pi u_n(\delta_n)^2 \tau_n}{(1-\tau_n)}} \\ &= \delta_n^2 \left( \sup_H E_0 \right)^{\tau_n} (\pi e)^{1-\tau_n} e^{\frac{4\pi u_n(\delta_n)^2}{1-\tau_n}}. \end{aligned}$$

Lemma 2.1 yields

$$\delta_n^2 e^{4\pi \frac{u_n(\delta_n)^2}{1-\tau_n}} \leq 1,$$

therefore

$$\int_{D_{\delta_n}} e^{4\pi u_n^2} dx \leq \left( \sup_H E_0 \right)^{\tau_n} (\pi e)^{1-\tau_n}.$$

Since  $\tau_n \rightarrow 0$ , we obtain the conclusion by taking the lim sup as  $n \rightarrow \infty$  on both sides.  $\square$

In order to prove Theorem 1.1 on  $H_{rad}$  for  $\alpha = 0$ , it is sufficient to show that, if  $u_n \rightharpoonup 0$ , there exists a sequence  $\delta_n$  satisfying the hypotheses of Lemma 2.2 and such that

$$\int_{D_{\delta_n}} (e^{4\pi u_n^2} - 1) dx \longrightarrow 0. \quad (24)$$

Note that, by dominated convergence theorem, (24) holds if there exists  $f \in L^1(D)$  such that

$$e^{4\pi u_n^2} \leq f \quad (25)$$

in  $D \setminus D_{\delta_n}$ . In the next lemma we will chose a function  $f \in L^1(D)$  with critical growth near 0 (i.e.  $f(x) \approx \frac{1}{|x|^2 \log^2 |x|}$ ) and define  $\delta_n$  so that (25) is satisfied.

**Lemma 2.3.** *Take  $u_n \in H_{rad}$  such that*

$$\sup_{D \setminus D_r} u_n \longrightarrow 0 \quad \forall r \in (0, 1). \quad (26)$$

*Then there exists a sequence  $\delta_n \in (0, 1)$  such that*

1.  $\delta_n \longrightarrow 0$ .
2.  $\tau_n := \int_{D_{\delta_n}} |\nabla u_n|^2 dx \longrightarrow 0$ .
3.  $\int_{D \setminus D_{\delta_n}} e^{4\pi u_n^2} dx \longrightarrow \pi$ .

*Proof.* We consider the function

$$f(x) := \begin{cases} \frac{1}{|x|^2 \log^2 |x|} & |x| \leq e^{-1} \\ e^2 & |x| \in (e^{-1}, 1]. \end{cases} \quad (27)$$

Note that  $f \in L^1(D)$  and

$$\inf_{(0,1)} f = e^2. \quad (28)$$

Let us fix  $\gamma_n \in (0, \frac{1}{n})$  such that  $\int_{D_{\gamma_n}} |\nabla u_n|^2 dx \leq \frac{1}{n}$ . We define

$$\tilde{\delta}_n := \inf \left\{ r \in (0, 1) : e^{4\pi u_n^2(x)} \leq f(x) \text{ for } r \leq |x| \leq 1 \right\} \in [0, 1).$$

and

$$\delta_n := \begin{cases} \tilde{\delta}_n & \text{if } \tilde{\delta}_n > 0 \\ \gamma_n & \text{if } \tilde{\delta}_n = 0. \end{cases}$$

By definition we have

$$e^{4\pi u_n^2} \leq f \quad \text{in } D \setminus D_{\delta_n},$$

thus 3 follows by dominated convergence Theorem. To conclude the proof it suffices to prove that if  $n_k \nearrow +\infty$  is chosen so that  $\delta_{n_k} = \tilde{\delta}_{n_k} \forall k$ , then

$$\lim_{k \rightarrow \infty} \delta_{n_k} = \lim_{k \rightarrow \infty} \tau_{n_k} = 0. \quad (29)$$



For such  $n_k$  one has

$$e^{4\pi u_{n_k}(\delta_{n_k})^2} = f(\delta_{n_k}). \quad (30)$$

In particular using (28) we obtain

$$e^{4\pi u_{n_k}(\delta_{n_k})^2} = f(\delta_{n_k}) \geq e^2 > 1$$

which, by (26), yields  $\delta_{n_k} \xrightarrow{k \rightarrow \infty} 0$ . Finally, Lemma 2.1 and (30) imply

$$1 \geq \delta_{n_k}^{2(1-\tau_{n_k})} e^{4\pi u_{n_k}(\delta_{n_k})^2} = \frac{\delta_{n_k}^{-2\tau_{n_k}}}{\log^2 \delta_{n_k}}$$

so that  $\tau_{n_k} \xrightarrow{k \rightarrow \infty} 0$  (otherwise the limit of the RHS would be  $+\infty$ ).  $\square$

Combining Lemma 2.2 with Lemma 2.3 we immediately get (6) for radially symmetric functions:

**Proposition 2.1.** *Let  $u_n \in H_{rad}$  and  $\alpha \in (-1, +\infty]$ . If for any  $r \in (0, 1)$*

$$\sup_{D \setminus D_r} u_n \longrightarrow 0,$$

*then*

$$\limsup_{n \rightarrow \infty} E_\alpha(u_n) \leq \frac{\pi(1+e)}{(1+\alpha)}.$$

*Proof.* If  $\alpha = 0$  the proof follows directly applying Lemma 2.3 and Lemma 2.2. If  $\alpha \neq 0$  consider

$$v_n(x) = (1+\alpha)^{\frac{1}{2}} u_n(|x|^{\frac{1}{1+\alpha}}).$$

We have

$$\int_D |\nabla v_n|^2 dx = \int_D |\nabla u_n|^2 dx$$

and hence  $v_n \in H_{rad}$ . Moreover we compute

$$\int_D |x|^{2\alpha} e^{(1+\alpha)u_n^2} dx = \frac{1}{1+\alpha} \int_D e^{4\pi v_n^2} dx$$

and the claim follows at once from the case  $\alpha = 0$ .  $\square$

To pass from Proposition 2.1 to Theorem 1.1 we will use symmetric rearrangements. We recall that given a measurable function  $u : \mathbb{R}^2 \rightarrow [0, +\infty)$ , the symmetric decreasing rearrangement of  $u$  is the unique right-continuous radially symmetric and decreasing function  $u^* : \mathbb{R}^2 \rightarrow [0, +\infty)$  such that

$$|\{u > t\}| = |\{u^* > t\}| \quad \forall t > 0.$$

Among the properties of  $u^*$  we recall that

1. If  $u \in L^p(\mathbb{R}^2)$ , then  $u^* \in L^p(\mathbb{R}^2)$  and  $\|u^*\|_p = \|u\|_p$ .

2. If  $u \in H_0^1(D)$ , then  $u^* \in H_{0,rad}^1(D)$  and

$$\int_D |\nabla u^*|^2 dx \leq \int_D |\nabla u|^2 dx. \quad (31)$$

3. If  $u, v : \mathbb{R}^2 \rightarrow [0, +\infty)$ , then

$$\int_{\mathbb{R}^2} u^*(x)v^*(x)dx \geq \int_{\mathbb{R}^2} u(x)v(x)dx. \quad (32)$$

In particular if  $u \in H_0^1(D)$  and  $\alpha \leq 0$ ,

$$\int_D |x|^{2\alpha} e^{u^*} dx \geq \int_D |x|^{2\alpha} e^u dx. \quad (33)$$

Note that (33) does not hold if  $\alpha > 0$ . We refer the reader to [11] for a more detailed introduction to symmetric rearrangements.

*Proof of Theorem 1.1.* Take  $u_n \in H$  such that  $u_n \rightarrow 0$  and let  $u_n^*$  be the symmetric decreasing rearrangement of  $u_n$ . Then  $u_n^* \in H_{rad}$  and since  $\|u_n^*\|_2 = \|u_n\|_2 \rightarrow 0$  we have  $\sup_{D \setminus D_r} u_n^* \rightarrow 0$   $\forall r > 0$ . Thus from (33) and Proposition 2.1 we get

$$\limsup_{n \rightarrow \infty} E_\alpha(u_n) \leq \limsup_{n \rightarrow \infty} E_\alpha(u_n^*) \leq \frac{\pi(1+e)}{1+\alpha}.$$

□

In the next section we will need the following local version of Theorem 1.1.

**Corollary 2.2.** Fix  $\delta > 0$ ,  $\alpha \in (-1, 0]$  and take  $u_n \in H_0^1(D_\delta)$  such that  $\int_{D_\delta} |\nabla u_n|^2 dx \leq 1$  and  $u_n \rightarrow 0$  in  $H_0^1(D_\delta)$ . For any choice of sequences  $\delta_n \rightarrow 0$ ,  $x_n \in \Omega$  such that  $D_{\delta_n}(x_n) \subset D_\delta$  we have

$$\limsup_{n \rightarrow \infty} \int_{D_{\delta_n}(x_n)} |x|^{2\alpha} e^{4\pi(1+\alpha)u_n^2} dx \leq \frac{\pi e}{1+\alpha} \delta^{2(1+\alpha)}.$$

*Proof.* Let us define  $\tilde{u}_n(x) := u_n(\delta x)$ . Note that  $\tilde{u}_n \in H$  and satisfies the hypotheses of Theorem 1.1, thus

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_{D_\delta} |x|^{2\alpha} (e^{4\pi u_n^2} - 1) dx \\ = \delta^{2(1+\alpha)} \limsup_{n \rightarrow \infty} \int_D |x|^{2\alpha} (e^{4\pi \tilde{u}_n^2} - 1) dx \\ \leq \delta^{2(1+\alpha)} \frac{\pi e}{1+\alpha}. \end{aligned}$$

Thus we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_{D_{\delta_n}(x_n)} |x|^{2\alpha} e^{4\pi(1+\alpha)u_n^2} dx \\ = \limsup_{n \rightarrow \infty} \int_{D_{\delta_n}(x_n)} |x|^{2\alpha} (e^{4\pi(1+\alpha)u_n^2} - 1) dx \\ \leq \int_{D_\delta} |x|^{2\alpha} (e^{4\pi u_n^2} - 1) dx \\ \leq \delta^{2(1+\alpha)} \frac{\pi e}{1+\alpha}. \end{aligned}$$

□

**Remark 2.2.** We remark that for  $\alpha \in (-1, 0]$  by Theorem 1.1 it is enough to show that

$$\sup_H E_\alpha > \frac{\pi(1+e)}{1+\alpha}$$

in order to prove existence of extremal functions for  $E_\alpha$  (see [4], [7]).

We conclude this section pointing out that as we just did for the Carleson-Chang type estimates, one can have a singular version of the Onofri inequality (14) (see Proposition A.2 in the Appendix). In particular one can deduce the following generalized version of Corollary 2.1.

**Corollary 2.3.**  $\forall \delta, \tau > 0, c \in \mathbb{R}$  and  $\alpha \in (-1, 0]$  we have

$$\int_{D_\delta} |x|^{2\alpha} e^{cu} dx \leq \frac{\pi e^{1+\frac{c^2\tau}{16\pi(1+\alpha)}} \delta^{2(1+\alpha)}}{1+\alpha}$$

$\forall u \in H_0^1(D_\delta)$  such that  $\int_{D_\delta} |\nabla u|^2 dx \leq \tau$ .

### 3 Extremals on Compact Surfaces: Notations and Preliminaries

Let  $(\Sigma, g)$  be a smooth, closed Riemannian surface. In this section, and in the rest of the paper, we will fix  $p_1, \dots, p_m \in \Sigma$  and consider a positive function  $h \in C^1(\Sigma \setminus \{p_1, \dots, p_m\})$  satisfying (18). More precisely, denoting by  $d$  the Riemannian distance on  $(\Sigma, g)$  and by  $B_r$  the corresponding metric ball, we will assume that for some  $\delta > 0$ ,

$$\frac{h}{d(\cdot, p_i)^{2\alpha_i}} \in C_+^1(B_\delta(p_i)) := \{f \in C^1(B_\delta(p_i)) : f > 0\} \quad \text{for } i = 1, \dots, m. \quad (34)$$

In order to distinguish the singular points  $p_1, \dots, p_m$  from the regular ones, we introduce a singularity index function

$$\alpha(x) := \begin{cases} \alpha_i & \text{if } x = p_i \\ 0 & x \in \Sigma \setminus \{p_1, \dots, p_m\}. \end{cases} \quad (35)$$

Clearly condition (34) implies that the limit

$$K(p) := \lim_{q \rightarrow p} \frac{h(q)}{d(q, p)^{2\alpha(p)}} \quad (36)$$

exists and is strictly positive for any  $p \in \Sigma$ . We will study functionals of the form (17) on the space

$$\mathcal{H} := \left\{ u \in H^1(\Sigma) : \int_\Sigma |\nabla u|^2 dv_g \leq 1, \int_\Sigma u dv_g = 0 \right\}.$$

To simplify the notation we will set

$$\bar{\alpha} := \min \left\{ 0, \min_{1 \leq i \leq m} \alpha_i \right\}$$

and

$$\bar{\beta} := 4\pi(1 + \bar{\alpha}).$$

Given  $s \geq 1$ , the symbols  $\|\cdot\|_s$ ,  $L^s(\Sigma)$  will denote the standard  $L^s$ -norm and  $L^s$ -space on  $\Sigma$  with respect to the metric  $g$ . Since we will deal with the singular metric  $g_h = gh$  we will also consider

$$\|u\|_{s,h} := \int_{\Sigma} |u|^s dv_{g_h} = \int_{\Sigma} h |u|^s dv_g$$

and

$$L^s(\Sigma, g_h) := \{u : \Sigma \longrightarrow \mathbb{R} \text{ Borel-measurable, } \|u\|_{s,h} < +\infty\}.$$

In this section we will prove the existence of an extremal function for  $E_{\Sigma,h}^{\beta,\lambda,q}$  for the subcritical case  $\beta < \bar{\beta}$ . We begin by stating some well known but useful Lemmas:

**Lemma 3.1.** *If  $u \in H^1(\Sigma)$  then  $e^{u^2} \in L^s(\Sigma) \cap L^s(\Sigma, g_h)$ ,  $\forall s \geq 1$ .*

*Proof.* From (34) we have  $h \in L^r(\Sigma)$  for some  $r > 1$ , hence it is sufficient to prove that  $e^{u^2} \in L^s(\Sigma)$ ,  $\forall s \geq 1$ . Moreover, since

$$e^{su^2} = e^{s(u-\bar{u})^2 + 2s(u-\bar{u})\bar{u} + s\bar{u}^2} \leq e^{2s(u-\bar{u})^2} e^{2s\bar{u}^2},$$

without loss of generality we can assume  $\bar{u} = 0$ . Take  $\varepsilon > 0$  such that  $2s\varepsilon \leq 4\pi$  and a function  $v \in C^1(\Sigma)$  satisfying  $\|\nabla_g(v-u)\|_2^2 \leq \varepsilon$  and  $\int_{\Sigma} v dv_g = 0$ . By (9), we have

$$\|e^{2s(u-v)^2}\|_1 + \|e^{2s\varepsilon \frac{u^2}{\|\nabla u\|_2^2}}\|_1 < +\infty. \quad (37)$$

Note that

$$e^{su^2} \leq e^{s(u-v)^2} e^{2suv}. \quad (38)$$

By (37), we have  $e^{s(u-v)^2} \in L^2(\Sigma)$  and, since  $v \in L^\infty(\Sigma)$ ,

$$e^{2suv} \leq e^{s\varepsilon \frac{u^2}{\|\nabla u\|_2^2}} e^{C(\varepsilon,s,\|\nabla u\|_2)v^2} \in L^2(\Sigma).$$

Hence using (38) and Holder's inequality we get  $e^{su^2} \in L^1(\Sigma)$ .  $\square$

**Lemma 3.2.** *If  $u_n \in \mathcal{H}$  and  $u_n \rightharpoonup u \neq 0$  weakly in  $H^1(\Sigma)$ , then*

$$\sup_n \int_{\Sigma} h e^{p\bar{\beta}u_n^2} dv_g < +\infty$$

$$\forall 1 \leq p < \frac{1}{1 - \|\nabla u\|_2^2}.$$

*Proof.* Observe that

$$e^{p\bar{\beta}u_n^2} \leq e^{p\bar{\beta}(u_n-u)^2} e^{2p\bar{\beta}u_n u}. \quad (39)$$

Since

$$\frac{1}{p} > 1 - \|\nabla u\|_2^2 \geq \|\nabla u_n\|_2^2 - \|\nabla u\|_2^2 = \|\nabla(u_n - u)\|_2^2 + o(1) \implies \limsup_{n \rightarrow \infty} \|\nabla(u_n - u)\|_2^2 < \frac{1}{p},$$

by (20) we get  $\|e^{p\bar{\beta}(u_n-u)^2}\|_{s,h} \leq C$  for some  $s > 1$ . Taking  $\frac{1}{s} + \frac{1}{s'} = 1$  and using Lemma 3.1 we have

$$e^{2ps'\bar{\beta}u_n u} \leq e^{\frac{\bar{\beta}}{2}u_n^2} e^{C_{s,\alpha,p}u^2} \in L^1(\Sigma, g_h) \implies \|e^{2p\bar{\beta}u_n u}\|_{s',h} \leq C.$$

Thus from (39) we get  $\|e^{p\bar{\beta}u_n^2}\|_{1,h} \leq C$ .  $\square$

Existence of extremals for  $\beta < \bar{\beta}$  is a simple consequence of Lemma 3.2 and Vitali's convergence Theorem.

**Lemma 3.3.**  $\forall \beta \in (0, \bar{\beta}), \lambda \in [0, \lambda_q(\Sigma, g)), q > 1$  we have

$$\sup_{\mathcal{H}} E_{\Sigma, h}^{\beta, \lambda, q} < +\infty$$

and the supremum is attained.

*Proof.* Let  $u_n \in \mathcal{H}$  be a maximizing sequence for  $E_{\Sigma, h}^{\beta, \lambda, q}$ , and assume  $u_n \rightharpoonup u$  weakly in  $H^1(\Sigma)$ . We claim that  $e^{\beta u_n^2(1 + \lambda \|u_n\|_q^2)}$  is uniformly bounded in  $L^p(\Sigma, g_h)$  for some  $p > 1$ . In particular by Vitali's convergence theorem we get  $E_{\Sigma, h}^{\beta, \lambda, q}(u_n) \rightarrow E_{\Sigma, h}^{\beta, \lambda, q}(u)$  with  $E_{\Sigma, h}^{\beta, \lambda, q}(u) < +\infty$ . Hence  $E_{\Sigma, h}^{\beta, \lambda, q}(u) = \sup_{\mathcal{H}} E_{\Sigma, h}^{\beta, \lambda, q}(u)$ , proving the conclusion. If  $u = 0$ , then

$$\beta(1 + \lambda \|u_n\|_q^2) \rightarrow \beta < \bar{\beta},$$

and the claim is proved taking  $1 < p < \frac{\bar{\beta}}{\beta}$  and using (20). If  $u \neq 0$ , since

$$(1 - \|\nabla u\|_2^2)(1 + \lambda \|u_n\|_q^2) \leq 1 - \|\nabla u\|_2^2 + \lambda \|u\|_q^2 + o(1) \leq 1 - (\lambda_q(\Sigma) - \lambda) \|u\|_q^2 + o(1) < 1$$

we can find  $p > 1$  such that  $\limsup_{n \rightarrow \infty} p(1 + \lambda \|u_n\|_q^2) < \frac{1}{1 - \|\nabla u\|_2^2}$ , and the claim follows from Lemma 3.2.  $\square$

The behaviour of extremal functions as  $\beta \rightarrow \bar{\beta}$  will be studied in the next section. As for now we can study the convergence of the suprema.

**Lemma 3.4.** As  $\beta \nearrow \bar{\beta}$  we have

$$\sup_{\mathcal{H}} E_{\Sigma, h}^{\beta, \lambda, q} \rightarrow \sup_{\mathcal{H}} E_{\Sigma, h}^{\bar{\beta}, \lambda, q}.$$

*Proof.* Clearly, since  $\beta < \bar{\beta}$ , we have

$$\limsup_{\beta \nearrow \bar{\beta}} \sup_{\mathcal{H}} E_{\Sigma, h}^{\beta, \lambda, q} \leq \sup_{\mathcal{H}} E_{\Sigma, h}^{\bar{\beta}, \lambda, q}.$$

On the other hand, by monotone convergence theorem we have

$$\liminf_{\beta \nearrow \bar{\beta}} \sup_{\mathcal{H}} E_{\Sigma, h}^{\beta, \lambda, q} \geq \liminf_{\beta \nearrow \bar{\beta}} E_{\Sigma, h}^{\beta, \lambda, q}(v) = E_{\Sigma, h}^{\bar{\beta}, \lambda, q}(v) \quad \forall v \in \mathcal{H},$$

which gives

$$\liminf_{\beta \nearrow \bar{\beta}} \sup_{\mathcal{H}} E_{\Sigma, h}^{\beta, \lambda, q} \geq \sup_{\mathcal{H}} E_{\Sigma, h}^{\bar{\beta}, \lambda, q}.$$

$\square$

We conclude this section with some Remarks concerning isothermal coordinates and Green's functions. We recall that, given any point  $p \in \Sigma$ , we can always find a small neighborhood  $\Omega$  of  $p$  and a local chart

$$\psi : \Omega \longrightarrow D_{\delta_0} \subset \mathbb{R}^2 \quad (40)$$

such that

$$\psi(p) = 0 \quad (41)$$

and

$$(\psi^{-1})^* g = e^\varphi |dx|^2 \quad (42)$$

where

$$\varphi \in C^\infty(\overline{D_{\delta_0}}) \quad \text{and} \quad \varphi(0) = 0. \quad (43)$$

For any  $\delta < \delta_0$  we will denote  $\Omega_\delta := \psi^{-1}(D_\delta)$ . More generally if  $D_r(x) \subseteq D_{\delta_0}$  we define  $\Omega_r(\psi^{-1}(x)) := \psi^{-1}(D_r(x))$ . We stress that (36) and (42) also imply

$$(\varphi^{-1})^* g_h = |x|^{2\alpha(p)} V(x) e^\varphi |dx|^2. \quad (44)$$

with

$$0 < V \in C^0(\overline{D_{\delta_0}}) \quad \text{and} \quad V(0) = K(p). \quad (45)$$

For any  $p \in \Sigma$  we denote  $G_p^\lambda$  the solution of

$$\begin{cases} -\Delta_g G_p^\lambda = \delta_p + \lambda \|G_p^\lambda\|_q^{2-q} |G_p^\lambda|^{q-2} G_p^\lambda - \frac{1}{|\Sigma|} \left( 1 + \lambda \|G_p^\lambda\|_q^{2-q} \int_\Sigma |G_p^\lambda|^{q-2} G_p^\lambda dv_g \right) \\ \int_\Sigma G_p^\lambda dv_g = 0. \end{cases} \quad (46)$$

In local coordinates satisfying (40)-(45) we have

$$G_p^\lambda(\psi^{-1}(x)) = -\frac{1}{2\pi} \log |x| + A_p^\lambda + \xi(x) \quad (47)$$

with  $\xi \in C^1(\overline{D_{\delta_0}})$  and  $\xi(x) = O(|x|)$ . Observe that  $G_p^0$  is the standard Green's function for  $-\Delta_g$ .

**Lemma 3.5.** *As  $\lambda \rightarrow 0$  we have  $G_p^\lambda \rightarrow G_p^0$  in  $L^s(\Sigma) \forall s$  and  $A_p^\lambda \rightarrow A_p^0$ .*

*Proof.* Let us denote  $c_\lambda := \frac{\lambda}{|\Sigma|} \|G_p^\lambda\|_q^{2-q} \int_\Sigma |G_p^\lambda|^{q-2} G_p^\lambda dv_g$ . Observe that

$$-\Delta_g (G_p^\lambda - G_p^0) = \lambda \|G_p^\lambda\|_q^{2-q} |G_p^\lambda|^{q-2} G_p^\lambda - c_\lambda.$$

Since

$$\left\| \|G_p^\lambda\|_q^{2-q} |G_p^\lambda|^{q-2} G_p^\lambda \right\|_{\frac{q}{q-1}} = \|G_p^\lambda\|_q$$

by elliptic estimates we find

$$\|G_p^\lambda - G_p^0\|_\infty \leq \|G_p^\lambda - G_p^0\|_{W^{2, \frac{q}{q-1}}(\Sigma)} \leq C\lambda \|G_p^\lambda\|_q. \quad (48)$$

In particular

$$\|G_p^\lambda\|_q \leq \|G_p^0\|_q + \|G_p^\lambda - G_p^0\|_q \leq \|G_p^0\|_q + C\lambda \|G_p^\lambda\|_\infty \leq \|G_p^0\|_q + C\lambda \|G_p^\lambda\|_q,$$

hence for sufficiently small  $\lambda$  we have

$$\|G_p^\lambda\|_q \leq C\|G_p^0\|_q.$$

Thus by (48), as  $\lambda \rightarrow 0$  we find

$$\|G_p^\lambda - G_p^0\|_\infty \rightarrow 0.$$

In particular  $G_p^\lambda \rightarrow G_p^0$  in  $L^s$  for any  $s > 1$ . Since  $A_p^\lambda - A_p^0 = (G_p^\lambda - G_p^0)(p)$  we also get  $A_p^\lambda \rightarrow A_p^0$ .  $\square$

**Lemma 3.6.** Fix  $p \in \Sigma$  and let  $(\Omega, \psi)$  be a local chart satisfying (40)-(45). As  $\delta \rightarrow 0$  we have

$$\int_{\Sigma \setminus \Omega_\delta} |\nabla G_p^\lambda|^2 dv_g = -\frac{1}{2\pi} \log \delta + A_p^\lambda + \lambda \|G_p^\lambda\|_q^2 + O(\delta |\log \delta|).$$

*Proof.* Integrating by parts we have

$$\int_{\Sigma \setminus \Omega_\delta} |\nabla G_p^\lambda|^2 dv_g = - \int_{\Sigma \setminus \Omega_\delta} \Delta_g G_p^\lambda G_p^\lambda dv_g - \int_{\partial \Omega_\delta} G_p^\lambda \frac{\partial G_p^\lambda}{\partial \nu} d\sigma_g. \quad (49)$$

For the first term, using the definition of  $G_p^\lambda$  we get

$$\begin{aligned} - \int_{\Sigma \setminus \Omega_\delta} \Delta_g G_p^\lambda G_p^\lambda dv_g &= \lambda \|G_p^\lambda\|_q^{2-q} \int_{\Sigma \setminus \Omega_\delta} |G_p^\lambda|^q dv_g - \left( \frac{1}{|\Sigma|} + c_\lambda \right) \int_{\Sigma \setminus \Omega_\delta} G_p^\lambda dv_g \\ &= \lambda \|G_p^\lambda\|_q^2 + o(1). \end{aligned} \quad (50)$$

For the second term we use (47) to find

$$- \int_{\partial \Omega_\delta} G_p^\lambda \frac{\partial G_p^\lambda}{\partial \nu} d\sigma_g = -\frac{1}{2\pi} \log \delta + A_p^\lambda + O(\delta |\log \delta|). \quad (51)$$

$\square$

## 4 Blow-up Analysis for the Critical Exponent.

In this section we will study the critical case  $\beta = \bar{\beta}$ .

Let us fix  $q > 1, \lambda \in [0, \lambda_q(\Sigma, g))$  and take a sequence  $\beta_n \nearrow \bar{\beta}$  ( $\beta_n < \bar{\beta}$  for any  $n \in \mathbb{N}$ ). To simplify the notation we will set  $E_n := E_{\Sigma, h}^{\beta_n, \lambda, q}$ . By Lemma 3.3, for any  $n$  we can take a function  $u_n \in \mathcal{H}$  such that

$$E_n(u_n) = \sup_{\mathcal{H}} E_n. \quad (52)$$

Up to subsequences, we can always assume that

$$u_n \rightharpoonup u_0 \quad \text{in } H^1(\Sigma) \quad (53)$$

and

$$u_n \rightarrow u_0 \quad \text{in } L^s(\Sigma) \quad \forall s \geq 1. \quad (54)$$

**Lemma 4.1.** *If  $u_0 \neq 0$ , then*

$$E_n(u_n) \longrightarrow E_{\Sigma,h}^{\bar{\beta},\lambda,q}(u_0) < +\infty. \quad (55)$$

*In particular*

$$\sup_{\mathcal{H}} E_{\Sigma,h}^{\bar{\beta},\lambda,q} < +\infty$$

*and  $u_0$  is an extremal function.*

*Proof.* If  $u_0 \neq 0$  we can argue as in Lemma 3.3 to find  $p > 1$  such that  $e^{\beta_n u_n^2(1+\lambda\|u_n\|_q^2)}$  is uniformly bounded in  $L^p(\Sigma, g_h)$ . Vitali's convergence Theorem yields (55). Lemma 3.4 implies

$$\sup_{\mathcal{H}} E_{\Sigma,h}^{\bar{\beta},\lambda,q} = E_{\Sigma,h}^{\bar{\beta},\lambda,q}(u_0) < +\infty.$$

□

Thus it is sufficient to study the case  $u_0 = 0$ , which we will assume for the rest of this section. In the same spirit of Theorem 1.1 and (7), we will prove the following sharp upper bound for  $E_n(u_n)$ .

**Proposition 4.1.** *If  $u_0 = 0$  we have*

$$\limsup_{n \rightarrow \infty} E_n(u_n) \leq \frac{\pi e}{1 + \bar{\alpha}} \max_{p \in \Sigma, \alpha(p) = \bar{\alpha}} K(p) e^{\bar{\beta} A_p^\lambda} + |\Sigma|_{g_h}$$

where  $A_p^\lambda$  is defined as in (47) and  $|\Sigma|_{g_h} := \int_{\Sigma} h \, dv_g$ .

**Remark 4.1.** *We remark that the quantity*

$$\max_{p \in \Sigma, \alpha(p) = \bar{\alpha}} K(p) e^{\bar{\beta} A_p^\lambda}$$

*is well defined. Indeed if  $\bar{\alpha} < 0$  the set of points such that  $\alpha(p) = \bar{\alpha}$  is finite. On the other hand if  $\bar{\alpha} = 0$  we have that  $K \equiv h$  on  $\Sigma \setminus \{p_1, \dots, p_m\} = \{p \in \Sigma : \alpha(p) = \bar{\alpha}\}$  and the function  $h e^{\bar{\beta} A_p^\lambda}$  is continuous on  $\Sigma$  and has zeros at the points  $p_1, \dots, p_m$ .*

*In particular Lemma 4.1 and Proposition 4.1 give a proof of an Adimurthi-Druet type inequality, namely*

$$\sup_{\mathcal{H}} E_{\Sigma,h}^{\bar{\beta},\lambda,q} < +\infty.$$

The rest of this section is devoted to the proof of Proposition 4.1.

**Lemma 4.2.** *There exists  $s > 1$  such that  $u_n \in \mathcal{H} \cap W^{2,s}(\Sigma) \, \forall \, n$ . Moreover  $\|\nabla u_n\|_2 = 1$  and we have*

$$-\Delta_g u_n = \gamma_n h(x) u_n e^{b_n u_n^2} + s_n(x) \quad (56)$$

*where*

$$b_n := \beta_n(1 + \lambda\|u_n\|_q^2) \longrightarrow \bar{\beta}, \quad (57)$$

$$\limsup_n \gamma_n < +\infty \quad \text{and} \quad \gamma_n \int_{\Sigma} h u_n^2 e^{u_n^2} dv_g \longrightarrow 1, \quad (58)$$



and

$$s_n := \lambda_n \|u_n\|_q^{2-q} |u_n|^{q-2} u_n - c_n \quad (59)$$

with

$$\lambda_n \longrightarrow \lambda, \quad (60)$$

and

$$c_n := \frac{1}{|\Sigma|} \left( \gamma_n \int_{\Sigma} u_n e^{b_n u_n^2} dv_{g_h} + \lambda_n \|u_n\|_q^{2-q} \int_{\Sigma} |u_n|^{q-2} u_n dv_g \right). \quad (61)$$

In particular we have

$$c_n \longrightarrow 0, \quad \|s_n\|_{\frac{q}{q-1}} \longrightarrow 0 \quad (62)$$

as  $n \rightarrow +\infty$ .

*Proof.* The maximality of  $u_n$  clearly implies  $\|\nabla u_n\|_2 = 1$ . Using Langrange multipliers theorem, it is simple to verify that  $u_n$  satisfies

$$-\Delta_g u_n = \nu_n b_n h(x) u_n e^{b_n u_n^2} + \lambda \nu_n \beta_n \mu_n \|u_n\|_q^{2-q} |u_n|^{q-2} u_n - c_n. \quad (63)$$

where  $b_n$  is defined as in (57),  $\mu_n := \int_{\Sigma} h u_n^2 e^{b_n u_n^2} dv_g$ ,

$$c_n := \frac{1}{|\Sigma|} \left( \gamma_n \int_{\Sigma} h u_n e^{b_n u_n^2} dv_g + \lambda \nu_n \beta_n \mu_n \|u_n\|_q^{2-q} \int_{\Sigma} |u_n|^{q-2} u_n dv_g \right), \quad (64)$$

and  $\nu_n \in \mathbb{R}$ . We define  $\gamma_n := \nu_n b_n$ ,  $\lambda_n := \lambda \nu_n \beta_n \mu_n$  and  $s_n(x) := \lambda_n \|u_n\|_q^{2-q} |u_n|^{q-2} u_n - c_n$  so that (56) and (59) are satisfied. Observe also that

$$\left\| \|u_n\|_q^{2-q} |u_n|^{q-2} u_n \right\|_{\frac{q}{q-1}} = \|u_n\|_q \longrightarrow 0. \quad (65)$$

and

$$\|u_n\|_q^{2-q} \left| \int_{\Sigma} |u_n|^{q-2} u_n dv_g \right| \leq \|u_n\|_q |\Sigma|^{\frac{1}{q}} \longrightarrow 0 \quad (66)$$

If  $s_0 > 1$  is such that  $h \in L^{s_0}(\Sigma)$ , using Lemma 3.1 and standard Elliptic regularity, we find  $u_n \in W^{2,s}(\Sigma) \forall 1 < s < \min\{s_0, \frac{q}{q-1}\}$ . Multiplying (63) by  $u_n$  and integrating on  $\Sigma$  we get

$$1 = \nu_n b_n \mu_n + \lambda \nu_n \beta_n \mu_n \|u_n\|_q^2 = \nu_n b_n \mu_n \left( 1 + \frac{\lambda \beta_n \|u_n\|_q^2}{b_n} \right) = \gamma_n \mu_n (1 + o(1))$$

from which we get the second part of (58). As a consequence we also have

$$\lambda_n = \lambda \nu_n \beta_n \mu_n = \lambda \gamma_n \mu_n \frac{\beta_n}{b_n} \longrightarrow \lambda. \quad (67)$$

Now we prove  $\limsup_{n \rightarrow \infty} \gamma_n < +\infty$  or, equivalently,  $\liminf_{n \rightarrow \infty} \mu_n > 0$ . For any  $t > 0$ , we have

$$E_n(u_n) \leq \frac{1}{t^2} \int_{\{|u_n| > t\}} h u_n^2 e^{b_n u_n^2} dv_g + \int_{\{|u_n| \leq t\}} h e^{b_n u_n^2} dv_g \leq \frac{1}{t^2} \int_{\Sigma} h u_n^2 e^{b_n u_n^2} dv_g + |\Sigma|_{g_h} + o(1)$$

from which

$$\liminf_{n \rightarrow \infty} \mu_n = \liminf_{n \rightarrow \infty} \int_{\Sigma} h u_n^2 e^{b_n u_n^2} dv_g \geq t^2 \left( \sup_{\mathcal{H}} E_{\Sigma, h}^{\bar{\beta}, \lambda, q} - |\Sigma|_{g_h} \right) > 0.$$

It remains to prove that  $c_n \rightarrow 0$  which, with (65) and (67), completes the proof of (62). For any  $t > 0$

$$\gamma_n \int_{\Sigma} h|u_n| e^{b_n u_n^2} dv_g \leq \frac{\gamma_n}{t} \int_{\{|u_n| > t\}} h u_n^2 e^{b_n u_n^2} dv_g + \gamma_n \int_{\{|u_n| \leq t\}} h|u_n| e^{b_n u_n^2} dv_g = \frac{1 + o(1)}{t} + o(1).$$

Since  $t$  can be taken arbitrarily large we find

$$\gamma_n \int_{\Sigma} h u_n e^{b_n u_n^2} dv_g \rightarrow 0.$$

Combined with (61), (66) and (67), this yields  $c_n \rightarrow 0$ .  $\square$

By Lemma 4.2 we know that  $u_n \in C^0(\Sigma)$ , thus we can take a sequence  $p_n$  such that

$$m_n := \max_{\Sigma} |u_n| = u_n(p_n), \quad (68)$$

where the last equality holds up to changing the sign of  $u_n$ . Clearly if  $\sup_n m_n < +\infty$ , then we would have  $E_n(u_n) \rightarrow |\Sigma|_{g_h}$  which contradicts Lemma 3.4. Thus, up to subsequences, we will assume

$$m_n \rightarrow +\infty \quad \text{and} \quad p_n \rightarrow p. \quad (69)$$

**Lemma 4.3.** *Let  $\Omega \subset \Sigma$  be an open subset such that*

$$\limsup_{n \rightarrow +\infty} \|\nabla u_n\|_{L^2(\Omega)} < 1.$$

*Then*

$$\|u_n\|_{L_{loc}^{\infty}(\Omega)} \leq C.$$

*Proof.* Fix  $\tilde{\Omega} \Subset \Omega$ . Take a cut-off function  $\xi \in C_0^{\infty}(\Omega)$  such that  $0 \leq \xi \leq 1$  and  $\xi \equiv 1$  in  $\Omega'$  where  $\tilde{\Omega} \Subset \Omega' \Subset \Omega$ . Since

$$\begin{aligned} \int_{\Sigma} |\nabla u_n \xi|^2 dv_g &= \int_{\Omega} |\nabla u_n|^2 \xi^2 dv_g + 2 \int_{\Omega} u_n \xi \nabla u_n \cdot \nabla \xi dv_g + \int_{\Omega} |\nabla \xi|^2 u_n^2 dv_g \leq \\ &\leq (1 + \varepsilon) \int_{\Omega} |\nabla u_n|^2 \xi^2 dv_g + C_{\varepsilon} \int_{\Omega} |\nabla \xi|^2 u_n^2 dv_g \end{aligned}$$

and  $\varepsilon$  can be taken arbitrarily small, we find

$$\limsup_{n \rightarrow \infty} \|\nabla(u_n \xi)\|_{L^2(\Sigma)}^2 < 1.$$

Thus, applying (20) to  $v_n := \frac{\xi u_n}{\|\nabla(\xi u_n)\|_{L^2(\Sigma)}}$  we find

$$\left\| e^{\bar{\beta} u_n^2 (1 + \lambda \|u_n\|_q^2)} \right\|_{L^{s_0}(\Omega', g_h)} \leq C \quad (70)$$

for some  $s_0 > 1$ . From (62) and (70),  $-\Delta_g u_n$  is uniformly bounded in  $L^s(\Omega')$  for any  $s < \min\{s_0, \frac{q}{q-1}\}$ . If we take another cut-off function  $\tilde{\xi} \in C_0^{\infty}(\Omega')$  such that  $\tilde{\xi} \equiv 1$  in  $\tilde{\Omega}$ , applying elliptic estimates to  $\tilde{\xi} u_n$  in  $\Omega'$  we find  $\sup_{\Omega'} \tilde{\xi} u_n \leq C$  and hence  $\sup_{\tilde{\Omega}} u_n \leq C$ .  $\square$

From Lemma 4.3 one can deduce that  $|\nabla u_n|^2 \rightharpoonup \delta_p$ , that is  $u_n$  concentrates at  $p$ . Intuitively it is natural to expect that concentration for maximizing sequences happens in the regions in which  $h$  is larger. We will show that  $p$  must be a minimum point of the singularity index  $\alpha$  defined in (35). This will clarify the difference between the cases  $\overline{\alpha} < 0$  and  $\overline{\alpha} = 0$ : in the former, the blow-up point  $p$  will be one of the singular points  $p_1, \dots, p_m$ , while in the latter  $p \in \Sigma \setminus \{p_1, \dots, p_m\}$  (cfr. Remark 4.2 and Proposition 4.3). The next step consists in studying the behaviour of  $u_n$  around  $p$ . Arguing as in [13] we will prove that a suitable scaling of  $u_n$  converges to a solution of a (possibly singular) Liouville-type equation on  $\mathbb{R}^2$  (see Proposition 4.2).

Again we consider a local chart  $(\Omega, \psi)$  satisfying (40)-(45). From now on we will denote  $x_n := \psi(p_n)$  and

$$v_n = u_n \circ \psi^{-1}. \quad (71)$$

Define  $t_n$  and  $\tilde{t}_n$  so that

$$t_n^{2(1+\alpha(p))} \gamma_n m_n^2 e^{b_n m_n^2} = 1, \quad (72)$$

$$\tilde{t}_n^2 |x_n|^{2\alpha(p)} \gamma_n m_n^2 e^{b_n m_n^2} = 1. \quad (73)$$

**Lemma 4.4.** *For any  $\beta < \overline{\beta}$  we have*

$$t_n^{2(1+\alpha(p))} m_n^2 e^{\beta m_n^2} \rightarrow 0, \quad \tilde{t}_n^2 |x_n|^{2\alpha(p)} m_n^2 e^{\beta m_n^2} \rightarrow 0$$

as  $n \rightarrow +\infty$ . In particular, for any  $s \geq 0$  we have

$$\lim_{n \rightarrow +\infty} t_n m_n^s = 0, \quad \lim_{n \rightarrow +\infty} \tilde{t}_n m_n^s = 0.$$

Moreover as  $n \rightarrow +\infty$  we have

$$\frac{|x_n|}{t_n} \rightarrow +\infty \iff \frac{|x_n|}{\tilde{t}_n} \rightarrow +\infty. \quad (74)$$

*Proof.* Since the result can be proven both for  $t_n$  and  $\tilde{t}_n$  with the same argument, we will prove it here only for  $t_n$ . By (57), (58) and (72)

$$\begin{aligned} t_n^{2(1+\alpha(p))} m_n^2 e^{\beta m_n^2} &= \frac{e^{(\beta-b_n)m_n^2}}{\gamma_n} = e^{(\beta-b_n)m_n^2} \int_{\Sigma} h u_n^2 e^{b_n u_n^2} dv_g (1 + o(1)) \\ &\leq \int_{\Sigma} h u_n^2 e^{\beta u_n^2} dv_g (1 + o(1)). \end{aligned}$$

Take  $s = \frac{\overline{\beta}'}{\beta}$  (i.e.  $1/s + \beta/\overline{\beta} = 1$ ) and  $s_0 > 1$  such that  $h \in L^{s_0}(\Sigma)$ . Then

$$\int_{\Sigma} h u_n^2 e^{\beta u_n^2} dv_g \leq \|u_n^2\|_{s,h} \|e^{\overline{\beta} u_n^2}\|_{1,h}^{\frac{\beta}{\overline{\beta}}} \leq C \|h\|_{s_0}^{\frac{1}{s}} \|u_n^2\|_{ss_0'} \rightarrow 0.$$

As for the last claim it is enough to observe that from (72) and (73) one computes

$$\frac{|x_n|}{\tilde{t}_n} = \left( \frac{|x_n|}{t_n} \right)^{1+\alpha(p)}.$$

□

We define now

$$r_n := \begin{cases} \tilde{t}_n & \text{if } \frac{|x_n|}{t_n} \rightarrow +\infty \text{ as } n \rightarrow +\infty, \\ t_n & \text{otherwise} \end{cases} \quad (75)$$

and

$$\eta_n(x) := m_n (v_n(x_n + r_n x) - m_n) \quad (76)$$

where the function  $\eta_n$  is defined in  $D_{\frac{\delta_0}{r_n}}$ .

**Proposition 4.2.** *Up to subsequences,  $\eta_n \rightarrow \eta_0$  in  $C_{loc}^0(\mathbb{R}^2) \cap H_{loc}^1(\mathbb{R}^2)$ . Moreover*

(i) *if  $\frac{|x_n|}{r_n} \rightarrow +\infty$  as  $n \rightarrow +\infty$  the function  $\eta_0$  solves*

$$-\Delta \eta_0 = V(0)e^{2\bar{\beta}\eta_0} \quad (77)$$

$$\int_{\mathbb{R}^2} V(0)e^{2\bar{\beta}\eta_0} dx = 1; \quad (78)$$

(ii) *if  $\frac{|x_n|}{r_n} \rightarrow \bar{x}$  the function  $\eta_0$  solves*

$$-\Delta \eta_0 = |x + \bar{x}|^{2\alpha(p)} V(0)e^{2\bar{\beta}\eta_0} \quad (79)$$

$$\int_{\mathbb{R}^2} |x + \bar{x}|^{2\alpha(p)} V(0)e^{2\bar{\beta}\eta_0} dx = 1. \quad (80)$$

*Proof.* If  $\frac{|x_n|}{t_n} \rightarrow +\infty$  as  $n \rightarrow +\infty$  then  $r_n = \tilde{t}_n$  and it follows that  $\eta_n$  as in (76) satisfies

$$\begin{aligned} -\Delta \eta_n &= m_n r_n^2 e^{\varphi(x_n + r_n x)} \left( \gamma_n |x_n + r_n x|^{2\alpha(p)} V(x_n + r_n x) e^{b_n v_n^2} v_n(x_n + r_n x) + s_n(x_n + r_n x) \right) = \\ &= e^{\varphi(x_n + r_n x)} \left( \left| \frac{x_n}{|x_n|} + \frac{r_n}{|x_n|} x \right|^{2\alpha(p)} V(x_n + r_n x) \left( 1 + \frac{\eta_n}{m_n^2} \right) e^{b_n \left( 2\eta_n + \frac{\eta_n^2}{m_n^2} \right)} + m_n r_n^2 s_n(x_n + r_n x) \right). \end{aligned}$$

Otherwise we have that  $r_n = t_n$  and, up to subsequences,  $\frac{|x_n|}{t_n} \rightarrow \bar{x}$  as  $n \rightarrow +\infty$  and  $\eta_n$  satisfies

$$\begin{aligned} -\Delta \eta_n &= m_n r_n^2 e^{\varphi(x_n + r_n x)} \left( \gamma_n |x_n + r_n x|^{2\alpha(p)} V(x_n + r_n x) e^{b_n u_n^2} v_n(x_n + r_n x) + s_n(x_n + r_n x) \right) = \\ &= e^{\varphi(x_n + r_n x)} \left( \left| \frac{x_n}{r_n} + x \right|^{2\alpha(p)} V(x_n + r_n x) \left( 1 + \frac{\eta_n}{m_n^2} \right) e^{b_n \left( 2\eta_n + \frac{\eta_n^2}{m_n^2} \right)} + m_n r_n^2 s_n(x_n + r_n x) \right). \end{aligned}$$

Observe that from Lemma 4.4 and (62) we have

$$\begin{aligned} \int_{D_L} |m_n r_n^2 s_n(x_n + r_n x)|^{\frac{q}{q-1}} dx &= m_n^{\frac{q}{q-1}} r_n^{\frac{2}{q-1}} \int_{D_{Lr_n}(x_n)} |s_n(x)|^{\frac{q}{q-1}} dx \\ &\leq m_n^{\frac{q}{q-1}} r_n^{\frac{2}{q-1}} \|s_n\|_{\frac{q}{q-1}}^{\frac{q}{q-1}} \rightarrow 0 \end{aligned} \quad (81)$$

for any  $L > 0$ . Since  $2\eta_n + \frac{\eta_n^2}{m_n^2} \leq 0$  and  $|\eta_n| \leq 2m_n^2$ , for any  $L > 0$ , in both cases (i) and (ii) we can find  $s > 1$  such that

$$\| -\Delta\eta_n \|_{L^s(D_L)} \leq C.$$

Moreover  $\eta_n(0) = 0$ , thus we can exploit Sobolev's embeddings Theorems and Harnack's inequality to find a uniform bound for  $\eta_n$  in  $C^{0,\alpha}(D_{\frac{L}{2}})$  for any  $L > 0$ . Hence with a diagonal argument, we find a subsequence of  $\eta_n$  such that  $\eta \rightarrow \eta_0$  in  $H_{loc}^1(\mathbb{R}^2) \cap C_{loc}^0(\mathbb{R}^2)$ . Moreover  $\eta_0$  is a solution of (77) or (79), depending on our choice of  $r_n$ . It remains to prove (78) and (80) respectively. In order to do this we observe that in case (i) we compute

$$\begin{aligned} 1 &= - \int_{\Sigma} \Delta_g u_n u_n dv_g = \gamma_n \int_{\Sigma} h u_n^2 e^{b_n u_n^2} dv_g + \lambda_n \|u_n\|_q^2 \\ &\geq \gamma_n \int_{\Omega_{Lr_n}(p_n)} h u_n^2 e^{b_n u_n^2} dv_g + o(1) \\ &= V(0) \int_{D_L} e^{2\bar{\beta}\eta_0} dx + o(1). \end{aligned} \tag{82}$$

In particular it holds (see for instance [6])

$$\lim_{L \rightarrow +\infty} V(0) \int_{D_L} e^{2\bar{\beta}\eta_0} dx = \frac{1}{1 + \bar{\alpha}} \geq 1 \tag{83}$$

where the last inequality follows from the fact that  $\bar{\alpha} \leq 0$ . Hence with (82) we obtain (78). Similarly, in case (ii) we have

$$1 = - \int_{\Sigma} \Delta_g u_n u_n dv_g \geq V(0) \int_{D_L} |x + \bar{x}|^{2\alpha(p)} e^{2\bar{\beta}\eta_0} dx + o(1). \tag{84}$$

On the other hand (cfr. [22])

$$\lim_{L \rightarrow +\infty} V(0) \int_{D_L} |x + \bar{x}|^{2\alpha(p)} e^{2\bar{\beta}\eta_0} dx = \frac{1 + \alpha(p)}{1 + \bar{\alpha}} \geq 1 \tag{85}$$

where now the last inequality follows from the minimality of  $\bar{\alpha}$ . Therefore (80) is proven.  $\square$

**Remark 4.2.** From the proof of Proposition 4.2 it follows that if  $\bar{\alpha} < 0$  then by (82) and (83) we have that only case (ii) is possible. Moreover from (84) and (85) we get  $\alpha(p) = \bar{\alpha}$ , that is  $p$  must be one of the singular points  $p_1, \dots, p_m$ .

We stress that Proposition 4.2 gives us information on the nature of the point  $p$  only in the case  $\bar{\alpha} < 0$ . To have a deeper understanding of the case  $\bar{\alpha} = 0$  and a more complete analysis of the blow-up behaviour of  $u_n$  near the point  $p$  we will need few more steps (see Proposition 4.3).

**Lemma 4.5.** We have

$$(i) \lim_{L \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_{\Omega_{Lr_n}(p_n)} \gamma_n m_n h u_n e^{b_n u_n^2} dv_g = 1$$

$$(ii) \lim_{L \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_{\Omega_{Lr_n}(p_n)} \gamma_n h u_n^2 e^{b_n u_n^2} dv_g = 1$$

$$(iii) \lim_{L \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_{\Omega_{Lr_n}(p_n)} h e^{b_n u_n^2} dv_g = \limsup_{n \rightarrow +\infty} \frac{1}{\gamma_n m_n^2}.$$

*Proof.* Both (i) and (ii) follow easily from Proposition 4.2. We are left with the proof of (iii). By Proposition 4.2, for any  $L > 0$  we have

$$\lim_{n \rightarrow +\infty} \gamma_n m_n^2 \int_{\Omega_{Lr_n}(p_n)} h e^{b_n u_n^2} dv_g = 1 + o_L(1)$$

where  $o_L(1) \rightarrow 0$  as  $L \rightarrow \infty$ . Hence

$$\limsup_{n \rightarrow \infty} \frac{1}{\gamma_n m_n^2} = (1 + o_L(1)) \limsup_{n \rightarrow \infty} \int_{\Omega_{Lr_n}(p_n)} h e^{b_n u_n^2} dv_g$$

and we can conclude the proof letting  $L \rightarrow +\infty$ .  $\square$

Following [13], for any  $A > 1$  we define

$$u_n^A := \min\{u_n, \frac{m_n}{A}\}.$$

**Lemma 4.6.** *For any  $A > 1$  we have*

$$\limsup_{n \rightarrow \infty} \int_{\Sigma} |\nabla u_n^A|^2 dv_g = \frac{1}{A}.$$

*Proof.* Integrating by parts we have

$$\liminf_{n \rightarrow \infty} \int_{\Sigma} |\nabla u_n^A|^2 dv_g = \liminf_{n \rightarrow \infty} \int_{\Sigma} \nabla u_n^A \cdot \nabla u_n dv_g = \liminf_{n \rightarrow +\infty} - \int_{\Sigma} \Delta_g u_n u_n^A dv_g.$$

Fix now  $L > 0$ . By Proposition 4.2, for sufficiently large  $n$ ,  $\Omega_{Lr_n}(p_n) \subseteq \{u_n > \frac{m_n}{A}\}$ , hence using (56) and (59) we find

$$- \int_{\Sigma} \Delta_g u_n u_n^A dv_g = \gamma_n \int_{\Sigma} h u_n e^{b_n u_n^2} u_n^A dv_g + o(1) \geq \frac{\gamma_n m_n}{A} \int_{\Omega_{Lr_n}(p_n)} h u_n e^{b_n u_n^2} dv_g + o(1).$$

Hence passing to the limit as  $n, L \rightarrow +\infty$  we obtain

$$\liminf_{n \rightarrow \infty} \int_{\Sigma} |\nabla u_n^A|^2 dv_g = \liminf_{n \rightarrow +\infty} - \int_{\Sigma} \Delta_g u_n u_n^A dv_g \geq \frac{1}{A} \quad (86)$$

where the last inequality follows from Lemma 4.5. Similarly

$$- \int_{\Sigma} \Delta_g u_n \left(u_n - \frac{m_n}{A}\right)^+ dv_g \geq \gamma_n \int_{\Omega_{Lr_n}(p_n)} h u_n e^{b_n u_n^2} \left(u_n - \frac{m_n}{A}\right)^+ dv_g + o(1)$$

we get

$$\liminf_{n \rightarrow \infty} \int_{\Sigma} |\nabla (u_n - \frac{m_n}{A})^+|^2 dv_g \geq \frac{A-1}{A}, \quad (87)$$

again from Lemma 4.5. Clearly  $u_n = u_n^A + (u_n - \frac{m_n}{A})^+$  and  $\int_{\Sigma} \nabla u_n^A \cdot \nabla (u_n - \frac{m_n}{A})^+ dv_g = 0$  thus

$$1 = \int_{\Sigma} |\nabla u_n|^2 dv_g = \int_{\Sigma} |\nabla u_n^A|^2 dv_g + \int_{\Sigma} |\nabla (u_n - \frac{m_n}{A})^+|^2 dv_g$$

and from (86) and (87) we find

$$\lim_{n \rightarrow \infty} \int_{\Sigma} |\nabla u_n^A|^2 dv_g = \frac{1}{A} \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{\Sigma} |\nabla (u_n - \frac{m_n}{A})^+|^2 dv_g = \frac{A-1}{A}.$$

$\square$

With Lemma 4.6 we have a first rough version of Proposition 4.1.

**Lemma 4.7.**

$$\limsup_{n \rightarrow \infty} E_n(u_n) \leq \lim_{L \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_{\Omega_{Lr_n}(p_n)} h e^{b_n u_n^2} dv_g + |\Sigma|_{g_h}.$$

*Proof.* For any  $A > 1$  we have

$$E_n(u_n) = \int_{\{u_n \geq \frac{mn}{A}\}} h e^{b_n u_n^2} dv_g + \int_{\{u_n \leq \frac{mn}{A}\}} h e^{b_n (u_n^A)^2} dv_g.$$

By (58),

$$\int_{\{u_n \geq \frac{mn}{A}\}} h e^{b_n u_n^2} dv_g \leq \frac{A^2}{m_n^2} \int_{\Sigma} h u_n^2 e^{b_n u_n^2} dv_g = \frac{A^2}{\gamma_n m_n^2} (1 + o(1)).$$

For the last integral we apply Lemma 4.6. Since  $\limsup_{n \rightarrow \infty} \|\nabla u_n^A\|_2^2 \leq \frac{1}{A} < 1$ , (20) implies that  $e^{b_n (u_n^A)^2}$  is uniformly bounded in  $L^s(\Sigma, g_h)$  for some  $s > 1$ . Thus by Vitali's Theorem

$$\int_{\{u_n \leq \frac{mn}{A}\}} h e^{b_n (u_n^A)^2} dv_g \leq \int_{\Sigma} h e^{b_n (u_n^A)^2} dv_g \longrightarrow |\Sigma|_{g_h}.$$

Therefore we proved

$$\limsup_{n \rightarrow \infty} E_n(u_n) \leq \limsup_{n \rightarrow \infty} \frac{A^2}{\gamma_n m_n^2} + |\Sigma|_{g_h}.$$

As  $A \rightarrow 1$  we get the conclusion thanks to Lemma 4.5. □

**Lemma 4.8.** *We have*

$$\gamma_n m_n h u_n e^{b_n u_n^2} \rightharpoonup \delta_p$$

*weakly as measures as  $n \rightarrow +\infty$ .*

*Proof.* Take  $\xi \in C^0(\Sigma)$ . For  $L > 0$ ,  $A > 1$  we have

$$\begin{aligned} & \gamma_n m_n \int_{\Sigma} h u_n e^{b_n u_n^2} \xi dv_g \\ &= \gamma_n m_n \int_{\Omega_{Lr_n}(p_n)} h u_n e^{b_n u_n^2} \xi dv_g \\ &+ \gamma_n m_n \int_{\{u_n > \frac{mn}{A}\} \setminus \Omega_{Lr_n}(p_n)} h u_n e^{b_n u_n^2} \xi dv_g \\ &+ \gamma_n m_n \int_{\{u_n \leq \frac{mn}{A}\}} h u_n e^{b_n u_n^2} \xi dv_g \\ &=: I_n^1 + I_n^2 + I_n^3. \end{aligned}$$

We have

$$I_n^1 = \gamma_n m_n \int_{\Omega_{Lr_n}(p_n)} h u_n e^{b_n u_n^2} (\xi - \xi(p)) dv_g + \gamma_n m_n \int_{\Omega_{Lr_n}(p_n)} h u_n e^{b_n u_n^2} \xi(p) dv_g.$$

From  $\|\xi - \xi(p)\|_{L^\infty(\Omega_{Lr_n}(p_n))} \rightarrow 0$  as  $n \rightarrow +\infty$  and Lemma 4.5 we have

$$\lim_{L \rightarrow \infty} \lim_{n \rightarrow \infty} I_n^1 = \xi(p).$$

Similarly, using (58)

$$\begin{aligned} |I_n^2| &\leq m_n \int_{\{u_n > \frac{m_n}{A}\} \setminus \Omega_{Lr_n}(p_n)} \gamma_n h u_n e^{b_n u_n^2} |\xi| dv_g \\ &\leq A \int_{\{u_n > \frac{m_n}{A}\} \setminus \Omega_{Lr_n}(p_n)} \gamma_n h u_n^2 e^{b_n u_n^2} |\xi| dv_g \\ &\leq A \|\xi\|_{L^\infty(\Sigma)} \left( 1 - \int_{\Omega_{Lr_n}(p_n)} \gamma_n h u_n^2 e^{b_n u_n^2} dv_g + o(1) \right). \end{aligned}$$

Therefore, from Lemma 4.5,

$$\lim_{L \rightarrow \infty} \lim_{n \rightarrow \infty} I_n^2 = 0.$$

For the last integral by Lemma 4.6 and (20) there exist  $s > 1, C > 0$  such that

$$\int_{\Sigma} h e^{s \bar{\beta}(u_n^A)^2} dv_g \leq C$$

thus

$$|I_n^3| \leq \gamma_n m_n \|\xi\|_{\infty} \int_{\Sigma} h |u_n| e^{b_n(u_n)^2} dv_g \leq \gamma_n m_n \|\xi\|_{\infty} \|u_n\|_{s',h} \|e^{\bar{\beta}(u_n)^2}\|_{s,h} = \gamma_n m_n o(1).$$

By (iii) in Lemma 4.5 and Lemma 4.7 we get that  $\gamma_n m_n \rightarrow 0$  and hence we find  $|I_n^3| \rightarrow 0$  which gives the conclusion.  $\square$

Let now  $G_p^\lambda$  be the Green's function defined in (46). Using Lemma 4.8 we obtain:

**Lemma 4.9.**  $m_n u_n \rightarrow G_p^\lambda$  in  $C_{loc}^0(\Sigma \setminus \{p\}) \cap H_{loc}^1(\Sigma \setminus \{p\}) \cap L^s(\Sigma) \forall s > 1$ .

*Proof.* First we observe that  $\|m_n u_n\|_q$  is uniformly bounded. If not we could consider the sequence  $w_n := \frac{u_n}{\|u_n\|_q}$  which satisfies

$$-\Delta_g w_n = \gamma_n h \frac{u_n}{\|u_n\|_q} e^{b_n u_n^2} + \frac{s_n}{\|u_n\|_q}$$

Arguing as in Lemma 4.8 one can prove that  $\|\gamma_n h m_n u_n e^{b_n u_n^2}\|_1 \leq C$  and hence it follows

$$\frac{\|\gamma_n h u_n e^{b_n u_n^2}\|_1}{\|u_n\|_q} = \frac{\|\gamma_n h m_n u_n e^{b_n u_n^2}\|_1}{\|m_n u_n\|_q} \rightarrow 0$$

as  $n \rightarrow +\infty$ . Moreover it is easy to check with (59) and (61) that

$$\|s_n\|_1 \leq C \|u_n\|_q$$



and we have a uniform bound for  $-\Delta_g w_n$  in  $L^1(\Sigma)$ . Therefore  $w_n$  is uniformly bounded in  $W^{1,s}(\Sigma)$  for any  $1 < s < 2$  (see [24] for a reference on open sets in  $\mathbb{R}^2$ ). The weak limit  $w$  of  $w_n$  will satisfy

$$\int_{\Sigma} \nabla w \cdot \nabla \varphi \, dv_g = \lambda \int_{\Sigma} |w|^{q-2} w \varphi \, dv_g$$

for any  $\varphi \in C^1(\Sigma)$  such that  $\int_{\Sigma} \varphi \, dv_g = 0$ . But, since  $\lambda < \lambda_q(\Sigma, g)$ , this implies  $w = 0$  which contradicts  $\|w_n\|_q = 1$ . Hence  $\|m_n u_n\|_q \leq C$ .

This implies that  $-\Delta_g(m_n u_n)$  is uniformly bounded in  $L^1(\Sigma)$  and, as before,  $m_n u_n$  is uniformly bounded in  $W^{1,s}(\Sigma)$  for any  $s \in (1, 2)$ . By Lemma 4.8 we have  $m_n u_n \rightharpoonup G_p^\lambda$  weakly in  $W^{1,s}(\Sigma)$ ,  $s \in (1, 2)$  and strongly in  $L^r$  for any  $r \geq 1$ .

From Lemma 4.3 we get  $|\nabla u_n|^2 \rightharpoonup \delta_p$  and  $u_n$  is uniformly bounded in  $L_{loc}^\infty(\Sigma \setminus \{p\})$ . This implies the boundedness of  $-\Delta_g(m_n u_n)$  in  $L_{loc}^s(\Sigma \setminus \{p\})$  for some  $s > 1$  which gives a uniform bound for  $m_n u_n$  in  $W_{loc}^{2,s}(\Sigma \setminus \{p\})$ . Then, by elliptic estimates, we get  $m_n u_n \rightarrow G_p^\lambda$  in  $H_{loc}^1(\Sigma \setminus \{p\}) \cap C_{loc}^0(\Sigma \setminus \{p\})$ .  $\square$

As we did in the proof of Theorem 1.1, in the next Proposition we will use an Onofri-type inequality (Corollary 2.3) to control the energy on a small scale.

**Proposition 4.3.** *We have  $\alpha(p) = \bar{\alpha}$  and for any  $L > 0$*

$$\limsup_{n \rightarrow \infty} \int_{\Omega_{Lr_n}(p_n)} h e^{b_n u_n^2} \, dv_g \leq \frac{\pi K(p) e^{1+\bar{\beta} A_p^\lambda}}{1 + \bar{\alpha}}.$$

*Proof.* Let us observe that

$$\begin{aligned} \int_{D_{Lr_n}(x_n)} |x|^{2\alpha(p)} e^{b_n v_n^2} \, dx &= \int_{D_{Lr_n}(x_n)} |x|^{2(\alpha(p)-\bar{\alpha})+2\bar{\alpha}} e^{b_n v_n^2} \, dx \\ &\leq (Lr_n)^{2(\alpha(p)-\bar{\alpha})} \int_{D_{Lr_n}(x_n)} |x|^{2\bar{\alpha}} e^{b_n v_n^2} \, dx. \end{aligned} \quad (88)$$

Fix  $\delta > 0$  and set  $\tau_n = \int_{\Omega_\delta} |\nabla u_n|^2 \, dv_g = \int_{D_\delta} |\nabla v_n|^2 \, dy$ . Observe that, by Lemma 4.9,

$$m_n^2(1 - \tau_n) = \int_{\Sigma \setminus \Omega_\delta} |\nabla G_p^\lambda|^2 \, dv_g + o(1), \quad (89)$$

and

$$m_n^2 \|u_n\|_q^2 = \|G_p^\lambda\|_q^2 + o(1). \quad (90)$$

Since by Lemma 3.6 we have

$$\int_{\Sigma \setminus \Omega_\delta} |\nabla G_p^\lambda|^2 \, dv_g = -\frac{1}{2\pi} \log \delta + O(1) \xrightarrow{\delta \rightarrow 0} +\infty, \quad (91)$$

for  $\delta$  sufficiently small, we obtain

$$\begin{aligned} \tau_n(1 + \lambda \|u_n\|_2^2) &= \left(1 - \frac{1}{m_n^2} \int_{\Sigma \setminus \Omega_\delta} |\nabla G_p^\lambda|^2 \, dv_g + o\left(\frac{1}{m_n^2}\right)\right) \left(1 + \frac{\lambda}{m_n^2} \|G_p^\lambda\|_q^2 + o\left(\frac{1}{m_n^2}\right)\right) \\ &= 1 - \frac{1}{m_n^2} \left(\int_{\Sigma \setminus \Omega_\delta} |\nabla G_p^\lambda|^2 \, dv_g - \lambda \|G_p^\lambda\|_q^2\right) + o\left(\frac{1}{m_n^2}\right) < 1. \end{aligned} \quad (92)$$

We denote  $d_n := \sup_{\partial D_\delta} v_n$  and  $w_n := (v_n - d_n)^+ \in H_0^1(D_\delta)$ . Applying Holder's inequality we have

$$\begin{aligned} \int_{D_{Lr_n}(x_n)} |x|^{2\bar{\alpha}} e^{b_n v_n^2} dx &= e^{b_n d_n^2} \int_{D_{Lr_n}(x_n)} |x|^{2\bar{\alpha}} e^{b_n w_n^2 + 2b_n d_n w_n} dx \\ &\leq e^{b_n d_n^2} \left( \int_{D_{Lr_n}(x_n)} |x|^{2\bar{\alpha}} e^{\beta_n \frac{w_n^2}{\tau_n}} dx \right)^{\tau_n(1+\lambda\|u_n\|_q^2)} \left( \int_{D_{Lr_n}(x_n)} |x|^{2\bar{\alpha}} e^{\frac{2b_n w_n d_n}{1-\tau_n(1+\lambda\|u_n\|_q^2)}} dx \right)^{1-\tau_n(1+\lambda\|u_n\|_q^2)}. \end{aligned} \quad (93)$$

Observe that, for  $n \rightarrow +\infty$ , we have that  $\frac{w_n}{\sqrt{\tau_n}} \rightarrow 0$  uniformly on  $D_\delta \setminus D_{\delta'}$  for any  $0 < \delta' < \delta$ . Thus applying Corollary 2.2 to the function  $\frac{w_n}{\sqrt{\tau_n}}$  with  $\delta_n = Lr_n$ , we find

$$\limsup_{n \rightarrow \infty} \int_{D_{Lr_n}(x_n)} |x|^{2\bar{\alpha}} e^{\beta_n \frac{w_n^2}{\tau_n}} dx \leq \frac{\pi e}{1 + \bar{\alpha}} \delta^{2(1+\bar{\alpha})}. \quad (94)$$

Using Corollary 2.3 we find

$$\begin{aligned} \int_{D_{Lr_n}(x_n)} |x|^{2\bar{\alpha}} e^{\frac{2b_n w_n d_n}{1-\tau_n(1+\lambda\|u_n\|_q^2)}} dx &\leq \int_{D_\delta} |x|^{2\bar{\alpha}} e^{\frac{2b_n w_n d_n}{1-\tau_n(1+\lambda\|u_n\|_q^2)}} dx \\ &\leq \frac{\pi e^{1 + \frac{4b_n^2 d_n^2 \tau_n}{16\pi(1+\bar{\alpha})(1-\tau_n(1+\lambda\|u_n\|_q^2)^2)}}}{1 + \bar{\alpha}} \delta^{2(1+\bar{\alpha})} \\ &\leq \frac{\pi e^{1 + \frac{b_n d_n^2 \tau_n(1+\lambda\|u_n\|_q^2)}{(1-\tau_n(1+\lambda\|u_n\|_q^2)^2)}}}{1 + \bar{\alpha}} \delta^{2(1+\bar{\alpha})}. \end{aligned}$$

Combining this with (88), (93) and (94), we find

$$\limsup_{n \rightarrow \infty} \int_{D_{Lr_n}(x_n)} |x|^{2\alpha(p)} e^{b_n v_n^2} dx \leq \frac{\pi e \delta^{2(1+\bar{\alpha})}}{1 + \bar{\alpha}} \limsup_{n \rightarrow \infty} (Lr_n)^{2(\alpha(p)-\bar{\alpha})} e^{\frac{b_n d_n^2}{1-\tau_n(1+\lambda\|u_n\|_q^2)}}. \quad (95)$$

Using (92) and Lemma 4.9,

$$\lim_{n \rightarrow \infty} \frac{b_n d_n^2}{1 - \tau_n(1 + \lambda\|u_n\|_q^2)} = \frac{\bar{\beta}(\sup_{\partial B_\delta} G_p^\lambda)^2}{\left( \int_{\Sigma \setminus \Omega_\delta} |\nabla G_p^\lambda|^2 dv_g - \lambda \|G_p^\lambda\|_q^2 \right)} =: H(\delta). \quad (96)$$

Notice that by Lemma 3.6 and (47) we find

$$H(\delta) = -2(1 + \bar{\alpha}) \log \delta + \bar{\beta} A_p^\lambda + o_\delta(1). \quad (97)$$

From (95), (96) we obtain

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \int_{\Omega_{Lr_n}(p_n)} h e^{b_n u_n^2} dv_g &= \limsup_{n \rightarrow \infty} \int_{D_{Lr_n}(x_n)} V(x) |x|^{2\alpha(p)} e^{b_n v_n^2} dx \\ &\leq \frac{K(p) \pi e \delta^{2(1+\bar{\alpha})}}{1 + \bar{\alpha}} e^{H(\delta)} \limsup_{n \rightarrow +\infty} (Lr_n)^{2(\alpha(p)-\bar{\alpha})}. \end{aligned} \quad (98)$$

If  $\alpha(p) > \bar{\alpha}$  we would have  $(Lr_n)^{2(\alpha(p)-\bar{\alpha})} \rightarrow 0$  as  $n \rightarrow +\infty$ . This would imply, using Lemma 4.7, that

$$\limsup_{n \rightarrow +\infty} E_n(u_n) \leq |\Sigma_{g_h}|,$$

which is a contradiction since  $u_n$  is a maximizing sequence. Hence necessary we have  $\alpha(p) = \bar{\alpha}$ . Therefore combining (96), (97) and (98) we get

$$\limsup_{n \rightarrow +\infty} \int_{\Omega_{Lr_n}(p_n)} h e^{b_n u_n^2} dv_g \leq \frac{K(p) \pi e \delta^{2(1+\bar{\alpha})}}{1+\bar{\alpha}} e^{H(\delta)} = \frac{K(p) \pi e^{1+\bar{\beta} A_p^\lambda + o_\delta(1)}}{1+\bar{\alpha}}.$$

□

*Proof of Proposition 4.1.* The proof follows at once from Lemma 4.7 and Proposition 4.3. □

## 5 Test Functions and Existence of Extremals.

By Proposition 4.1, in order to prove existence of extremals for  $E_{\Sigma, h}^{\bar{\beta}, \lambda, q}$  it suffices to show that the value

$$\frac{\pi e}{1+\bar{\alpha}} \max_{p \in \Sigma, \alpha(p)=\bar{\alpha}} K(p) e^{\bar{\beta} A_p^\lambda} + |\Sigma|_{g_h}.$$

is exceeded. In this section we will show that this is indeed the case if  $\lambda$  is small enough.

**Proposition 5.1.** *There exists  $\lambda_0 > 0$  such that  $\forall 0 \leq \lambda < \lambda_0$  one has*

$$\sup_{u \in \mathcal{H}} E_{\Sigma, h}^{\bar{\beta}, \lambda, q} > \frac{\pi e}{1+\bar{\alpha}} \max_{p \in \Sigma, \alpha(p)=\bar{\alpha}} K(p) e^{\bar{\beta} A_p^\lambda} + |\Sigma|_{g_h}$$

*Proof.* Let  $p \in \Sigma$  be such that  $\alpha(p) = \bar{\alpha}$  and

$$K(p) e^{\bar{\beta} A_p^\lambda} = \max_{q \in \Sigma, \alpha(q)=\bar{\alpha}} K(q) e^{\bar{\beta} A_q^\lambda}.$$

In local coordinates  $(\Omega, \psi)$  satisfying (40)-(45) we define

$$w_\varepsilon(x) := \begin{cases} c_\varepsilon - \frac{\log\left(1 + \left(\frac{|\psi(x)|}{\varepsilon}\right)^{2(1+\bar{\alpha})}\right) + L_\varepsilon}{\bar{\beta} c_\varepsilon} & x \in \Omega_{\gamma_\varepsilon \varepsilon} \\ \frac{G_p^\lambda - \eta_\varepsilon \xi}{c_\varepsilon} & x \in \Omega_{2\gamma_\varepsilon \varepsilon} \setminus \Omega_{\gamma_\varepsilon \varepsilon} \\ \frac{G_p^\lambda}{c_\varepsilon} & x \in \Sigma \setminus \Omega_{2\gamma_\varepsilon \varepsilon} \end{cases} \quad (99)$$

and

$$u_\varepsilon := \frac{w_\varepsilon}{\sqrt{1 + \frac{\lambda}{c_\varepsilon^2} \|G_p^\lambda\|_q^2}}$$

where  $c_\varepsilon, L_\varepsilon$  will be chosen later,  $\gamma_\varepsilon = |\log \varepsilon|^{\frac{1}{1+\bar{\alpha}}}$ ,  $\xi$  is defined as in (47) and  $\eta_\varepsilon$  is a cut-off function such that  $\eta_\varepsilon \equiv 1$  in  $\Omega_{\gamma_\varepsilon \varepsilon}$ ,  $\eta_\varepsilon \in C_0^\infty(\Omega_{2\gamma_\varepsilon \varepsilon})$  and  $\|\nabla \eta_\varepsilon\|_{L^\infty(\Sigma)} = O(\frac{1}{\gamma_\varepsilon \varepsilon})$ . In order to have  $u_\varepsilon \in H^1(\Sigma)$  we choose  $L_\varepsilon$  so that

$$\bar{\beta} c_\varepsilon^2 - L_\varepsilon = \log \left( \frac{1 + \gamma_\varepsilon^{2(1+\bar{\alpha})}}{\gamma_\varepsilon^{2(1+\bar{\alpha})}} \right) + \bar{\beta} A_p^\lambda - 2(1+\bar{\alpha}) \log \varepsilon. \quad (100)$$

Observe that

$$\int_{\Omega_{\gamma_\varepsilon \varepsilon}} |\nabla w_\varepsilon|^2 dv_g = \frac{1}{\bar{\beta} c_\varepsilon^2} \left( \log(1 + \gamma_\varepsilon^{2(1+\bar{\alpha})}) - 1 + O(|\log \varepsilon|^{-2}) \right). \quad (101)$$

Since  $\xi \in C^1(\overline{D_{\delta_0}})$  and  $\xi(x) = O(|x|)$  we have

$$\begin{aligned} \int_{\Omega_{2\gamma_\varepsilon \varepsilon} \setminus \Omega_{\gamma_\varepsilon \varepsilon}} |\nabla(\eta_\varepsilon \xi)|^2 dv_g &= \int_{\Omega_{2\gamma_\varepsilon \varepsilon} \setminus \Omega_{\gamma_\varepsilon \varepsilon}} |\nabla \eta_\varepsilon|^2 \xi^2 dv_g \\ &+ \int_{\Omega_{2\gamma_\varepsilon \varepsilon} \setminus \Omega_{\gamma_\varepsilon \varepsilon}} |\nabla \xi|^2 \eta_\varepsilon^2 dv_g + 2 \int_{\Omega_{2\gamma_\varepsilon \varepsilon} \setminus \Omega_{\gamma_\varepsilon \varepsilon}} \eta_\varepsilon \xi \nabla \eta_\varepsilon \cdot \nabla \xi dv_g \\ &= O((\gamma_\varepsilon \varepsilon)^2), \end{aligned}$$

and similarly

$$\int_{\Omega_{2\gamma_\varepsilon \varepsilon} \setminus \Omega_{\gamma_\varepsilon \varepsilon}} \nabla G_p^\lambda \cdot \nabla(\eta_\varepsilon \xi) dv_g = O(\gamma_\varepsilon \varepsilon).$$

By Lemma 3.6 we have

$$\begin{aligned} c_\varepsilon^2 \int_{\Sigma \setminus \Omega_{\gamma_\varepsilon \varepsilon}} |\nabla w_\varepsilon|^2 dv_g &= \int_{\Sigma \setminus \Omega_{\gamma_\varepsilon \varepsilon}} |\nabla G_p^\lambda|^2 + O(\gamma_\varepsilon \varepsilon) = \\ &= -\frac{1}{2\pi} \log \gamma_\varepsilon \varepsilon + A_p^\lambda + \lambda \|G_p^\lambda\|_q^2 + O(\gamma_\varepsilon \varepsilon |\log(\gamma_\varepsilon \varepsilon)|). \end{aligned}$$

Observe that  $\gamma_\varepsilon \varepsilon \log(\gamma_\varepsilon \varepsilon) = o(|\log \varepsilon|^{-2})$ , therefore we get

$$\int_{\Sigma} |\nabla w_\varepsilon|^2 dv_g = \frac{1}{\bar{\beta} c_\varepsilon^2} \left( -1 - 2(1 + \bar{\alpha}) \log \varepsilon + \bar{\beta} A_p^\lambda + \bar{\beta} \lambda \|G_p^\lambda\|_q^2 + O(|\log \varepsilon|^{-2}) \right).$$

If we chose  $c_\varepsilon$  so that

$$\bar{\beta} c_\varepsilon^2 = -1 - 2(1 + \bar{\alpha}) \log \varepsilon + \bar{\beta} A_p^\lambda + O(|\log \varepsilon|^{-2}), \quad (102)$$

then  $u_\varepsilon - \bar{u}_\varepsilon \in \mathcal{H}$ . Observe also that (100), (102) yield

$$L_\varepsilon = -1 + O(|\log \varepsilon|^{-2}). \quad (103)$$

and

$$2\pi c_\varepsilon^2 = |\log \varepsilon| + O(1). \quad (104)$$

Since  $0 \leq w_\varepsilon \leq O(c_\varepsilon)$  in  $\Omega_{\gamma_\varepsilon \varepsilon}$  we get

$$\int_{\Omega_{\gamma_\varepsilon \varepsilon}} w_\varepsilon dv_g = O(c_\varepsilon (\gamma_\varepsilon \varepsilon)^2) = o(|\log \varepsilon|^{-2}).$$

Moreover

$$\begin{aligned} \int_{\Sigma \setminus \Omega_{\gamma_\varepsilon \varepsilon}} w_\varepsilon dv_g &= \int_{\Sigma \setminus \Omega_{\gamma_\varepsilon \varepsilon}} \frac{G_p^\lambda}{c_\varepsilon} dv_g - \int_{\Omega_{2\gamma_\varepsilon \varepsilon} \setminus \Omega_{\gamma_\varepsilon \varepsilon}} \frac{\eta_\varepsilon \xi}{c_\varepsilon} dv_g \\ &= O\left(\frac{(\gamma_\varepsilon \varepsilon)^2 |\log(\gamma_\varepsilon \varepsilon)|}{c_\varepsilon}\right) + O\left(\frac{(\gamma_\varepsilon \varepsilon)^3}{c_\varepsilon}\right) \\ &= o(|\log \varepsilon|^{-2}) \end{aligned}$$

therefore

$$\overline{w}_\varepsilon = o(|\log \varepsilon|^{-2}) = o(c_\varepsilon^{-4}). \quad (105)$$

From (102), (103) and (105) it follows that in  $\Omega_{\gamma_\varepsilon \varepsilon}$  we have

$$\overline{\beta}(w_\varepsilon - \overline{w}_\varepsilon)^2 \geq \overline{\beta}c_\varepsilon^2 - 2L_\varepsilon - 2 \log \left( 1 + \left( \frac{|\psi(x)|}{\varepsilon} \right)^{2(1+\overline{\alpha})} \right) + o(c_\varepsilon^{-2}).$$

We have

$$c_\varepsilon^2 \|w_\varepsilon - \overline{w}_\varepsilon\|_q^2 \geq \left( \int_{\Sigma \setminus \Omega_{2\gamma_\varepsilon \varepsilon}} |G_p^\lambda - c_\varepsilon \overline{w}_\varepsilon|^q dv_g \right)^{\frac{2}{q}} \geq \|G_p^\lambda\|_q^2 + o(c_\varepsilon^{-2}),$$

where the last inequality follows from (99) and Bernoulli's inequality after splitting the integral on regions where  $|G_p^\lambda| \geq |c_\varepsilon \overline{w}_\varepsilon|$  and  $|G_p^\lambda| \leq |c_\varepsilon \overline{w}_\varepsilon|$ . Therefore we find

$$\begin{aligned} \frac{1}{1 + \frac{\lambda}{c_\varepsilon^2} \|G_p^\lambda\|_q^2} \left( 1 + \frac{\lambda \|w_\varepsilon - \overline{w}_\varepsilon\|_q^2}{1 + \frac{\lambda}{c_\varepsilon^2} \|G_p^\lambda\|_q^2} \right) &\geq \frac{1 + 2 \frac{\lambda}{c_\varepsilon^2} \|G_p^\lambda\|_q^2 + o(c_\varepsilon^{-4})}{\left( 1 + \frac{\lambda}{c_\varepsilon^2} \|G_p^\lambda\|_q^2 \right)^2} \\ &= 1 - \frac{\lambda^2 \|G_p^\lambda\|_q^4}{c_\varepsilon^4} + o(c_\varepsilon^{-4}). \end{aligned} \quad (106)$$

Therefore

$$\overline{\beta}(u_\varepsilon - \overline{u}_\varepsilon)^2 (1 + \lambda \|u_\varepsilon - \overline{u}_\varepsilon\|_q^2) \geq \overline{\beta}c_\varepsilon^2 - 2L_\varepsilon - 2 \log \left( 1 + \left( \frac{|\psi(x)|}{\varepsilon} \right)^{2(1+\overline{\alpha})} \right) - \frac{\overline{\beta}\lambda^2 \|G_p^\lambda\|_q^4}{c_\varepsilon^2} + o(c_\varepsilon^{-2}).$$

It follows that

$$\begin{aligned} &\int_{\Omega_{\gamma_\varepsilon \varepsilon}} h e^{\overline{\beta}(u_\varepsilon - \overline{u}_\varepsilon)^2 (1 + \lambda \|u_\varepsilon - \overline{u}_\varepsilon\|_q^2)} dv_g \\ &\geq \int_{D_{\gamma_\varepsilon \varepsilon}} |x|^{2\overline{\alpha}} (V(0) + O(\gamma_\varepsilon \varepsilon)) \frac{e^{\overline{\beta}c_\varepsilon^2 - 2L_\varepsilon - \frac{\overline{\beta}\lambda^2 \|G_p^\lambda\|_q^4}{c_\varepsilon^2} + o(c_\varepsilon^{-2})}}{\left( 1 + \left( \frac{|x|}{\varepsilon} \right)^{2(1+\overline{\alpha})} \right)^2} dx \\ &= \frac{\pi V(0) \varepsilon^{2(1+\overline{\alpha})} \gamma_\varepsilon^{2(1+\overline{\alpha})}}{(1+\overline{\alpha})(1+\gamma_\varepsilon^{2(1+\overline{\alpha})})} e^{\overline{\beta}c_\varepsilon^2 - 2L_\varepsilon - \frac{\overline{\beta}\lambda^2 \|G_p^\lambda\|_q^4}{c_\varepsilon^2} + o(c_\varepsilon^{-2})} (1 + O(\gamma_\varepsilon \varepsilon)) \\ &= \frac{\pi V(0) \varepsilon^{2(1+\overline{\alpha})}}{(1+\overline{\alpha})} e^{\overline{\beta}c_\varepsilon^2 - 2L_\varepsilon - \frac{\overline{\beta}\lambda^2 \|G_p^\lambda\|_q^4}{c_\varepsilon^2} + o(c_\varepsilon^{-2})} (1 + O(c_\varepsilon^{-4})). \end{aligned}$$

Using (102) and (103) we find

$$\overline{\beta}c_\varepsilon^2 - 2L_\varepsilon = -2(1+\overline{\alpha}) \log \varepsilon + 1 + \overline{\beta}A_p^\lambda + O(c_\varepsilon^{-4})$$

so that

$$\int_{\Omega_{\gamma_\varepsilon \varepsilon}} h e^{\overline{\beta}(u_\varepsilon - \overline{u}_\varepsilon)^2 (1 + \lambda \|u_\varepsilon - \overline{u}_\varepsilon\|_q^2)} = \frac{\pi V(0) e^{1+\overline{\beta}A_p^\lambda}}{(1+\overline{\alpha})} \left( 1 - \frac{\overline{\beta}\lambda^2 \|G_p^\lambda\|_q^4}{c_\varepsilon^2} + o(c_\varepsilon^{-2}) \right). \quad (107)$$

Finally, with (105) and (106), we observe that

$$\begin{aligned}
& \int_{\Sigma \setminus \Omega_{2\gamma\varepsilon}} h e^{\bar{\beta}(u_\varepsilon - \bar{u}_\varepsilon)^2 (1 + \lambda \|u_\varepsilon - \bar{u}_\varepsilon\|_q^2)} dv_g \\
& \geq \int_{\Sigma \setminus \Omega_{2\gamma\varepsilon}} h dv_g + \bar{\beta}(1 + \lambda \|u_\varepsilon - \bar{u}_\varepsilon\|_q^2) \int_{\Sigma \setminus \Omega_{2\gamma\varepsilon}} h (u_\varepsilon - \bar{u}_\varepsilon)^2 dv_g \\
& \geq |\Sigma|_{g_h} + O((\gamma_\varepsilon \varepsilon)^{2(1+\bar{\alpha})}) + \bar{\beta} \left( 1 - \frac{\lambda^2 \|G_p^\lambda\|_q^4}{c_\varepsilon^4} + o(c_\varepsilon^{-4}) \right) \int_{\Sigma \setminus \Omega_{2\gamma\varepsilon}} h (w_\varepsilon - \bar{w}_\varepsilon)^2 dv_g \\
& = |\Sigma|_{g_h} + \frac{\bar{\beta} \|G_p^\lambda\|_{L^2(\Sigma, g_h)}}{c_\varepsilon^2} + o(c_\varepsilon^{-2}).
\end{aligned} \tag{108}$$

Hence from (107) and (108) it follows that

$$E_{\Sigma, h}^{\bar{\beta}, \lambda, q}(u_\varepsilon - \bar{u}_\varepsilon) \geq \frac{\pi K(p)}{1 + \bar{\alpha}} e^{1 + \bar{\beta} A_p^\lambda} + |\Sigma|_{g_h} + \frac{\bar{\beta}}{c_\varepsilon^2} \left( \|G_p^\lambda\|_{L^2(\Sigma, g_h)} - \frac{\pi K(p) e^{1 + \bar{\beta} A_p^\lambda} \lambda^2 \|G_p^\lambda\|_q^4}{1 + \bar{\alpha}} \right) + o(c_\varepsilon^{-2}),$$

where we used that by definition  $K(p) = V(0)$ . By Lemma 3.5, we know that

$$\left( \|G_p^\lambda\|_{L^2(\Sigma, g_h)} - \frac{\pi K(p) e^{1 + \bar{\beta} A_p^\lambda} \lambda^2 \|G_p^\lambda\|_q^4}{1 + \bar{\alpha}} \right) \longrightarrow \|G_p^0\|_{L^2(\Sigma, g_h)} > 0$$

as  $\lambda \rightarrow 0$  thus for sufficiently small  $\lambda$  we get the conclusion.  $\square$

To finish the proof of Theorem 1.2 we have to treat the case  $\lambda > \lambda_q(\Sigma, g)$ . Will use a family of test functions similar to the one used in [17].

**Lemma 5.1.** *If  $\beta > \bar{\beta}$  or  $\beta = \bar{\beta}$  and  $\lambda > \lambda_q(\Sigma, g)$ , we have*

$$\sup_{\mathcal{H}} E_{\Sigma, h}^{\bar{\beta}, \lambda, q} = +\infty.$$

*Proof.* Take  $p \in \Sigma$  such that  $\alpha(p) = \bar{\alpha}$  and a local chart  $(\Omega, \psi)$  satisfying (40)-(45). Let us define  $v_\varepsilon : D_{\delta_0} \longrightarrow [0, +\infty)$ ,

$$v_\varepsilon(x) := \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\log \frac{\delta_0}{\varepsilon}} & |x| \leq \varepsilon \\ \frac{\log \frac{\delta_0}{|x|}}{\sqrt{\log \frac{\delta_0}{\varepsilon}}} & \varepsilon \leq |x| \leq \delta_0 \end{cases}$$

and

$$u_\varepsilon(x) := \begin{cases} v_\varepsilon(\psi(x)) & x \in \Omega \\ 0 & x \in \Sigma \setminus \Omega. \end{cases}$$

It is simple to verify that

$$\int_{\Sigma} |\nabla u_\varepsilon|^2 dv_g = \int_{D_{\delta_0}} |\nabla v_\varepsilon|^2 dx = 1,$$

thus  $u_\varepsilon - \bar{u}_\varepsilon \in \mathcal{H}$ . By direct computation one has

$$\bar{u}_\varepsilon = O \left( \left( \log \frac{1}{\varepsilon} \right)^{-\frac{1}{2}} \right), \tag{109}$$

hence in  $\Omega_\varepsilon$  we have

$$(u_\varepsilon - \bar{u}_\varepsilon)^2 = \frac{1}{2\pi} \log \left( \frac{\delta_0}{\varepsilon} \right) + O(1).$$

Thus if  $\beta > \bar{\beta}$  we have

$$\begin{aligned} E_{\Sigma, h}^{\beta, \lambda, q}(u_\varepsilon - \bar{u}_\varepsilon) &\geq E_{\Sigma, h}^{\beta, 0, q}(u_\varepsilon - \bar{u}_\varepsilon) \geq \int_{\Omega_\varepsilon} h e^{\beta(u_\varepsilon - \bar{u}_\varepsilon)^2} dv_g \geq \frac{C}{\varepsilon^{\frac{\beta}{2\pi}}} \int_{D_\varepsilon} |x|^{2\bar{\alpha}} dx = \\ &= \frac{C\pi}{1 + \bar{\alpha}} \varepsilon^{2(1+\bar{\alpha}) - \frac{\beta}{2\pi}} = \tilde{C} \varepsilon^{\frac{\bar{\beta} - \beta}{2\pi}} \longrightarrow +\infty \end{aligned}$$

as  $\varepsilon \longrightarrow 0$ . For the case  $\beta = \bar{\beta}$  and  $\lambda > \lambda_q(\Sigma, g)$  we take a function  $u_0 \in H^1(\Sigma)$  such that

$$\begin{cases} \|\nabla u_0\|_2^2 = \lambda_q(\Sigma, g) \|u_0\|_q^2 \\ \int_\Sigma u_0 dv_g = 0 \\ \|u_0\|_q^2 = 1. \end{cases} \quad (110)$$

This function  $u_0$  will also satisfy

$$-\Delta_g u_0 = \lambda_q \|u_0\|_q^{2-q} |u_0|^{q-2} u_0 - c$$

where

$$c = \frac{\lambda_q}{|\Sigma|} \|u_0\|_q^{2-q} \int_\Sigma |u_0|^{q-2} u_0 dv_g.$$

Let us take  $t_\varepsilon, r_\varepsilon \longrightarrow 0$  such that

$$t_\varepsilon^2 |\log \varepsilon| \longrightarrow +\infty, \quad \frac{r_\varepsilon}{\varepsilon} \longrightarrow +\infty \quad \text{and} \quad \frac{\log^2 r_\varepsilon}{t_\varepsilon^2 |\log \varepsilon|} \longrightarrow 0. \quad (111)$$

We define

$$w_\varepsilon := u_\varepsilon \eta_\varepsilon + t_\varepsilon u_0 \quad (112)$$

where  $\eta_\varepsilon \in C_0^\infty(\Omega_{2r_\varepsilon})$  is a cut-off function such that  $\eta_\varepsilon \equiv 1$  in  $\Omega_{r_\varepsilon}$ ,  $0 \leq \eta_\varepsilon \leq 1$  and  $|\nabla \eta_\varepsilon| = O(r_\varepsilon^{-1})$ . It is straightforward that

$$\bar{w}_\varepsilon = O(|\log \varepsilon|^{-\frac{1}{2}}). \quad (113)$$

Observe that

$$\|\nabla w_\varepsilon\|_2^2 = \int_\Sigma |\nabla(u_\varepsilon \eta_\varepsilon)|^2 dv_g + t_\varepsilon^2 \|\nabla u_0\|_2^2 + 2t_\varepsilon \int_\Sigma \nabla u_0 \cdot \nabla(u_\varepsilon \eta_\varepsilon) dv_g.$$

Using the definition of  $u_\varepsilon, \eta_\varepsilon$  and (111) we find

$$\int_\Sigma |\nabla \eta_\varepsilon|^2 u_\varepsilon^2 dv_g = O(r_\varepsilon^{-2}) \int_{\Omega_{2r_\varepsilon} \setminus \Omega_{r_\varepsilon}} u_\varepsilon^2 dv_g = O(|\log \varepsilon|^{-1} \log^2 r_\varepsilon) = o(t_\varepsilon^2)$$

and

$$\left| \int_\Sigma u_\varepsilon \eta_\varepsilon \nabla u_\varepsilon \cdot \nabla \eta_\varepsilon dv_g \right| \leq O(r_\varepsilon^{-1}) \int_{\Omega_{2r_\varepsilon} \setminus \Omega_{r_\varepsilon}} |\nabla u_\varepsilon| u_\varepsilon dv_g = O(|\log r_\varepsilon| |\log \varepsilon|^{-1}) = o(t_\varepsilon^2).$$

Thus

$$\|\nabla(u_\varepsilon \eta_\varepsilon)\|_2^2 = \int_\Sigma |\nabla u_\varepsilon|^2 \eta_\varepsilon^2 dv_g + o(t_\varepsilon^2) \leq 1 + o(t_\varepsilon^2).$$

Moreover (110) gives  $\|\nabla u_0\|_2^2 = \lambda_q$  and

$$\left| \int_\Sigma \nabla u_0 \cdot \nabla(u_\varepsilon \eta_\varepsilon) dv_g \right| = \left| \lambda_q \int_\Sigma (|u_0|^{q-2} u_0 - c) \eta_\varepsilon u_\varepsilon dv_g \right| = O(1) \int_\Sigma u_\varepsilon dv_g = O(|\log \varepsilon|^{-\frac{1}{2}}) = o(t_\varepsilon).$$

Hence we have

$$\|\nabla w_\varepsilon\|_2^2 \leq 1 + \lambda_q t_\varepsilon^2 + o(t_\varepsilon^2).$$

Furthermore by dominated convergence we have,

$$\|w_\varepsilon - \overline{w}_\varepsilon\|_q^2 \geq t_\varepsilon^2 \left( \int_{\Sigma \setminus \Omega_{2r_\varepsilon}} |u_0 - \frac{\overline{w}_\varepsilon}{t_\varepsilon}|^q dv_g \right)^{\frac{2}{q}} = t_\varepsilon^2 \|u_0\|_q^2 + o(t_\varepsilon^2) = t_\varepsilon^2 + o(t_\varepsilon^2),$$

thus we get

$$\frac{1}{\|\nabla w_\varepsilon\|_2^2} \left( 1 + \lambda \frac{\|w_\varepsilon - \overline{w}_\varepsilon\|_q^2}{\|\nabla w_\varepsilon\|_2^2} \right) \geq 1 + (\lambda - \lambda_q) t_\varepsilon^2 + o(t_\varepsilon^2).$$

Finally, using (113), in  $\Omega_\varepsilon$  we find

$$\begin{aligned} & \frac{4\pi(1+\overline{\alpha})(w_\varepsilon - \overline{w}_\varepsilon)^2}{\|\nabla w_\varepsilon\|_2^2} \left( 1 + \lambda \frac{\|w_\varepsilon - \overline{w}_\varepsilon\|_q^2}{\|\nabla w_\varepsilon\|_2^2} \right) \\ &= (2(1+\overline{\alpha})|\log \varepsilon| + O(1)) (1 + (\lambda - \lambda_q) t_\varepsilon^2 + o(t_\varepsilon^2)) \\ &= -2(1+\overline{\alpha}) \log \varepsilon + (\lambda - \lambda_q) t_\varepsilon^2 |\log \varepsilon| + O(1), \end{aligned}$$

so that

$$\begin{aligned} E_{\Sigma, h}^{\overline{\beta}, \lambda, q} \left( \frac{w_\varepsilon - \overline{w}_\varepsilon}{\|\nabla w_\varepsilon\|_2} \right) &\geq \int_{\Omega_\varepsilon} h e^{\frac{4\pi(1+\overline{\alpha})(w_\varepsilon - \overline{w}_\varepsilon)^2}{\|\nabla w_\varepsilon\|_2^2} \left( 1 + \lambda \frac{\|w_\varepsilon - \overline{w}_\varepsilon\|_q^2}{\|\nabla w_\varepsilon\|_2^2} \right)} dv_g \geq \\ &\geq c\varepsilon^{-2(1+\overline{\alpha})} e^{(\lambda - \lambda_q) t_\varepsilon^2 |\log \varepsilon| + O(1)} \int_{D_\varepsilon} |y|^{2\overline{\alpha}} dy = \tilde{c} e^{(\lambda - \lambda_q) t_\varepsilon^2 |\log \varepsilon|} \longrightarrow +\infty \end{aligned}$$

as  $\varepsilon \rightarrow 0$ . □

**Remark 5.1.** *If there exists a point  $p \in \Sigma$  such that  $\alpha(p) = \overline{\alpha}$  and  $u_0(p) \neq 0$ , then one can argue as in [17] to prove that,*

$$\sup_{\mathcal{H}} E_{\Sigma, h}^{\overline{\beta}, \lambda, q} = +\infty$$

*also for  $\lambda = \lambda_q(\Sigma, g_0)$ . This is always true if  $\overline{\alpha} = 0$ .*



## A Onofri-type Inequalities for Disks.

Let  $(\Sigma, g)$  be a smooth, closed Riemannian surface. As a consequence of (9) one gets

$$\log \left( \frac{1}{|\Sigma|} \int_{\Sigma} e^{u-\bar{u}} dv_g \right) \leq \frac{1}{16\pi} \int_{\Sigma} |\nabla_g u|^2 dv_g + C(\Sigma, g). \quad (114)$$

While it is well known that the coefficient  $\frac{1}{16\pi}$  is sharp, the optimal value of  $C(\Sigma, g)$  is harder to determine. For the special case of the standard Euclidean sphere  $(S^2, g_0)$ , Onofri ([21]) proved that  $C(S^2, g_0) = 0$  and gave a complete characterization of the extremal functions for (114).

**Proposition A.1** ([21]).  *$\forall u \in H^1(S^2)$  we have*

$$\log \left( \frac{1}{4\pi} \int_{S^2} e^{u-\bar{u}} dv_{g_0} \right) \leq \frac{1}{16\pi} \int_{S^2} |\nabla_{g_0} u|^2 dv_{g_0},$$

*with equality holding if and only if  $e^u g_0$  is a metric on  $S^2$  with positive constant Gaussian curvature, or, equivalently,  $u = \log |\det d\varphi| + c$  with  $c \in \mathbb{R}$  and  $\varphi : S^2 \rightarrow S^2$  a conformal diffeomorphism of  $S^2$ .*

We will prove now Proposition 1.1 by means of the stereographic projection.

*Proof.* Let us fix Euclidean coordinates  $(x_1, x_2, x_3)$  on  $S^2 \subseteq \mathbb{R}^3$  and denote  $N := (0, 0, 1)$  and  $S = (0, 0, -1)$  the north and the south pole. Let us consider the stereographic projection  $\pi : S^2 \setminus \{N\} \rightarrow \mathbb{R}^2$

$$\pi(x) := \left( \frac{x_1}{1-x_3}, \frac{x_2}{1-x_3} \right).$$

It is well known that  $\pi$  is a conformal diffeomorphism and

$$(\pi^{-1})^* g_0 = e^{u_0} |dx|^2 \quad (115)$$

where

$$u_0(x) = \log \left( \frac{4}{(1+|x|^2)^2} \right) \quad (116)$$

satisfies

$$-\Delta u_0 = 2e^{u_0} \quad \text{on } \mathbb{R}^2. \quad (117)$$

Given  $r > 0$ , let  $D_r := \{x \in \mathbb{R}^2 : |x| < r\}$  be the disk of radius  $r$  and  $S_r^2 = \pi^{-1}(D_r)$ . We consider the map  $T_r : H_0^1(D_r) \rightarrow H^1(S^2)$  defined by

$$T_r u(x) := \begin{cases} u(\pi(x)) - u_0(\pi(x)) & \text{on } S_r^2 \\ -2 \log \left( \frac{2}{1+r^2} \right) & \text{on } S^2 \setminus S_r^2. \end{cases}$$

Using (115) we find

$$\int_{S^2} e^{T_r u} dv_{g_0} \geq \int_{S_r^2} e^{T_r u} dv_{g_0} = \int_{D_r} e^{T_r u(\pi^{-1}(y))} e^{u_0} dy = \int_{D_r} e^{u(y)} dy. \quad (118)$$

Moreover, by (117),

$$\begin{aligned}
\int_{S_r^2} |\nabla T_r u|^2 dv_{g_0} &= \int_{D_r} |\nabla u|^2 dx - 2 \int_{D_r} \nabla u_0 \cdot \nabla u \, dy + \int_{D_r} |\nabla u_0|^2 dy \\
&= \int_{D_r} |\nabla u|^2 dy - 4 \int_{D_r} u e^{u_0} dy + \int_{D_r} |\nabla u_0|^2 dy \\
&= \int_{D_r} |\nabla u|^2 dy - 4 \int_{S_r^2} T_r u \, dv_{g_0} + \left( \int_{D_r} |\nabla u_0|^2 dy - 4 \int_{D_r} u_0 e^{u_0} dy \right).
\end{aligned}$$

It is easy to check with a direct computation that one has

$$\int_{D_r} |\nabla u_0|^2 dy = 16\pi \left( \log(1 + r^2) - \frac{r^2}{1 + r^2} \right)$$

and

$$\int_{D_r} u_0 e^{u_0} dy = 8\pi \log 2 - 8\pi + o(1),$$

where  $o(1) \rightarrow 0$  as  $r \rightarrow +\infty$ . Thus we get

$$\begin{aligned}
&\int_{S^2} |\nabla T_r u|^2 dv_{g_0} + 4 \int_{S^2} T_r u \, dv_{g_0} \\
&= \int_{D_r} |\nabla u|^2 dy + 16\pi (\log(1 + r^2) + 1 - 2\log 2 + o(1)).
\end{aligned} \tag{119}$$

Using (118), (119) and Proposition A.1 we can conclude

$$\begin{aligned}
\log \left( \frac{1}{\pi} \int_{D_r} e^u dy \right) &\leq \log \left( \frac{1}{\pi} \int_{S^2} e^{T_r u} dv_{g_0} \right) \\
&\leq \frac{1}{16\pi} \left( \int_{S^2} |\nabla T_r u|^2 dv_{g_0} + 2 \int_{S^2} T_r u \, dv_{g_0} \right) + 2\log 2 \\
&\leq \frac{1}{16\pi} \int_{D_r} |\nabla u|^2 dy + \log(1 + r^2) + 1 + o(1).
\end{aligned} \tag{120}$$

Now if  $u \in H_0^1(D)$  we can apply (120) to  $u_r(y) = u(\frac{y}{r})$  and since

$$\int_D e^u dx = \frac{1}{r^2} \int_{D_r} e^{u_r(y)} dy \quad \text{and} \quad \int_D |\nabla u|^2 dx = \int_{D_r} |\nabla u_r|^2 dy,$$

we find

$$\log \left( \frac{1}{\pi} \int_D e^u dx \right) \leq \frac{1}{16\pi} \int_D |\nabla u|^2 dx + 1 + o(1).$$

As  $r \rightarrow \infty$  we get the conclusion.  $\square$

As in [2], starting from (14) we can use a simple change of variables to obtain singular Onofri-type inequalities for the unit disk.

**Proposition A.2.** *Let  $-1 < \alpha \leq 0$ . Then for any  $u \in H_0^1(D)$  we have*

$$\log \left( \frac{1+\alpha}{\pi} \int_D |x|^{2\alpha} e^u dx \right) \leq \frac{1}{16\pi(1+\alpha)} \int_D |\nabla u|^2 dx + 1. \quad (121)$$

Moreover, if we restrict ourselves to the space  $H_{0,rad}^1(D)$ , we have that (121) holds true for any  $\alpha \in (-1, +\infty]$ .

*Proof.* As we did in the proof of Proposition 2.1, for  $u \in H_{0,rad}^1(D)$  we consider the function  $v(x) = u(|x|^{\frac{1}{1+\alpha}})$ , which is again in  $H_{0,rad}^1(D)$ . The second claim follows at once applying (14) to  $v$ . As for the first claim, if  $\alpha \leq 0$  we can use again symmetric rearrangements as we did in the proof of Theorem 1.1 to remove the symmetry assumption.  $\square$

Since

$$\int_D |x|^{2\alpha} dx = \frac{\pi}{1+\alpha},$$

Proposition A.2 can be written in a simpler form in terms of the singular metric  $g_\alpha = |x|^{2\alpha} |dx|^2$ .

**Corollary A.1.** *If  $u \in H_0^1(D)$  and  $-1 < \alpha \leq 0$  (or  $\alpha > 0$  and  $u \in H_{0,rad}^1(D)$ ), we have*

$$\log \left( \frac{1}{|D|_\alpha} \int_D e^u dv_{g_\alpha} \right) \leq \frac{1}{16\pi(1+\alpha)} \int_D |\nabla u|^2 dv_{g_\alpha} + 1$$

where  $|D|_\alpha = \frac{\pi}{(1+\alpha)}$  is the measure of  $D$  with respect to  $g_\alpha$ .

We stress that the constant 1 appearing in Proposition A.2 is sharp.

**Proposition A.3.** *For any  $-1 < \alpha \leq 0$  we have*

$$\inf_{u \in H_0^1(D)} \frac{1}{16\pi(1+\alpha)} \int_D |\nabla u|^2 dx - \log \left( \frac{1}{|D|_\alpha} \int_D |x|^{2\alpha} e^u dx \right) = -1.$$

Moreover, if we restrict ourselves to the space  $H_{0,rad}^1(D)$ , the conclusion above holds true for any  $\alpha \in (-1, +\infty)$ .

*Proof.* Let us denote

$$E_\alpha(u) := \frac{1}{16\pi(1+\alpha)} \int_D |\nabla u|^2 dx - \log \left( \frac{1}{|D|_\alpha} \int_D |x|^{2\alpha} e^u dv_g \right).$$

It is sufficient to exhibit a family of functions  $u_\varepsilon \in H_{0,rad}^1(D)$  such that  $E_\alpha(u_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} -1$ . Take  $\gamma_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} +\infty$  such that  $\varepsilon \gamma_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0$ , and define

$$u_\varepsilon(x) = \begin{cases} -2 \log \left( 1 + \left( \frac{|x|}{\varepsilon} \right)^{2(1+\alpha)} \right) + L_\varepsilon & \text{for } |x| \leq \gamma_\varepsilon \varepsilon \\ -4(1+\alpha) \log |x| & \text{for } \gamma_\varepsilon \varepsilon \leq |x| \leq 1 \end{cases}$$

where the quantity

$$L_\varepsilon := 2 \log \left( \frac{1 + \gamma_\varepsilon^{2(1+\alpha)}}{\gamma_\varepsilon^{2(1+\alpha)}} \right) - 4(1+\alpha) \log \varepsilon$$

is chosen so that  $u_\varepsilon \in H_0^1(D)$ . Simple computations show that

$$\frac{1}{16\pi(1+\alpha)} \int_D |\nabla u_\varepsilon|^2 dx = -1 - 2(1+\alpha) \log \varepsilon + o_\varepsilon(1)$$

and

$$\begin{aligned} \int_D |x|^{2\alpha} e^{u_\varepsilon} dx &= \frac{\varepsilon^{2(1+\alpha)} \gamma_\varepsilon^{2(1+\alpha)} e^{L_\varepsilon \pi}}{(1+\alpha)(1+\gamma_\varepsilon^{2(1+\alpha)})} + \frac{\pi}{1+\alpha} \left( \frac{1}{(\gamma_\varepsilon \varepsilon)^{2(1+\alpha)}} - 1 \right) \\ &= \frac{\pi \varepsilon^{-2(1+\alpha)}}{1+\alpha} (1 + o_\varepsilon(1)). \end{aligned}$$

Thus

$$E_\alpha(u_\varepsilon) \longrightarrow -1.$$

□

To conclude we remark that Propositions A.2 and A.3 can also be deduced directly using the singular versions of Proposition A.1 proved in [18], [19].

## References

- [1] Adimurthi and O. Druet. Blow-up analysis in dimension 2 and a sharp form of Trudinger-Moser inequality. *Comm. Partial Differential Equations*, 29(1-2):295–322, 2004.
- [2] Adimurthi and K. Sandeep. A singular Moser-Trudinger embedding and its applications. *NoDEA Nonlinear Differential Equations Appl.*, 13(5-6):585–603, 2007.
- [3] William Beckner. Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality. *Ann. of Math. (2)*, 138(1):213–242, 1993.
- [4] Lennart Carleson and Sun-Yung A. Chang. On the existence of an extremal function for an inequality of J. Moser. *Bull. Sci. Math. (2)*, 110(2):113–127, 1986.
- [5] Wen Xiong Chen. A Trüdinger inequality on surfaces with conical singularities. *Proc. Amer. Math. Soc.*, 108(3):821–832, 1990.
- [6] Wen Xiong Chen and Congming Li. Classification of solutions of some nonlinear elliptic equations. *Duke Math. J.*, 63(3):615–622, 1991.
- [7] G. Csatò and P. Roy. The singular moser-trudinger inequality on simply connected domains. *preprint*, <http://arxiv.org/pdf/1507.00323v1.pdf>, 2015.
- [8] G. Csatò and P. Roy. Extremal functions for the singular moser-trudinger inequality in 2 dimensions. *Calc. Var. Partial Differential Equations*, 10.1007/s00526-015-0867-5, to appear.
- [9] Martin Flucher. Extremal functions for the Trudinger-Moser inequality in 2 dimensions. *Comment. Math. Helv.*, 67(3):471–497, 1992.

- [10] Luigi Fontana. Sharp borderline Sobolev inequalities on compact Riemannian manifolds. *Comment. Math. Helv.*, 68(3):415–454, 1993.
- [11] S. Kesavan. *Symmetrization & applications*, volume 3 of *Series in Analysis*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.
- [12] Yu Xiang Li. Remarks on the extremal functions for the Moser-Trudinger inequality. *Acta Math. Sin. (Engl. Ser.)*, 22(2):545–550, 2006.
- [13] Yuxiang Li. Moser-Trudinger inequality on compact Riemannian manifolds of dimension two. *J. Partial Differential Equations*, 14(2):163–192, 2001.
- [14] Yuxiang Li. Extremal functions for the Moser-Trudinger inequalities on compact Riemannian manifolds. *Sci. China Ser. A*, 48(5):618–648, 2005.
- [15] Kai-Ching Lin. Extremal functions for Moser’s inequality. *Trans. Amer. Math. Soc.*, 348(7):2663–2671, 1996.
- [16] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. I. *Rev. Mat. Iberoamericana*, 1(1):145–201, 1985.
- [17] Guozhen Lu and Yunyan Yang. Sharp constant and extremal function for the improved Moser-Trudinger inequality involving  $L^p$  norm in two dimension. *Discrete Contin. Dyn. Syst.*, 25(3):963–979, 2009.
- [18] G. Mancini. Onofri-type inequalities for singular liouville equations. *Journal of Geometric Analysis*, 2015.
- [19] G. Mancini. Singular liouville equations on  $S^2$ : Sharp inequalities and existence results. *preprint*, 2015.
- [20] J. Moser. A sharp form of an inequality by N. Trudinger. *Indiana Univ. Math. J.*, 20:1077–1092, 1970/71.
- [21] E. Onofri. On the positivity of the effective action in a theory of random surfaces. *Comm. Math. Phys.*, 86(3):321–326, 1982.
- [22] J. Prajapat and G. Tarantello. On a class of elliptic problems in  $\mathbb{R}^2$ : symmetry and uniqueness results. *Proc. Roy. Soc. Edinburgh Sect. A*, 131(4):967–985, 2001.
- [23] Michael Struwe. Critical points of embeddings of  $H_0^{1,n}$  into Orlicz spaces. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 5(5):425–464, 1988.
- [24] Michael Struwe. Positive solutions of critical semilinear elliptic equations on non-contractible planar domains. *J. Eur. Math. Soc. (JEMS)*, 2(4):329–388, 2000.
- [25] Cyril Tintarev. Trudinger-Moser inequality with remainder terms. *J. Funct. Anal.*, 266(1):55–66, 2014.
- [26] Marc Troyanov. Prescribing curvature on compact surfaces with conical singularities. *Trans. Amer. Math. Soc.*, 324(2):793–821, 1991.

- [27] Neil S. Trudinger. On imbeddings into Orlicz spaces and some applications. *J. Math. Mech.*, 17:473–483, 1967.
- [28] Yunyan Yang. A sharp form of the Moser-Trudinger inequality on a compact Riemannian surface. *Trans. Amer. Math. Soc.*, 359(12):5761–5776 (electronic), 2007.
- [29] Yunyan Yang. Extremal functions for Trudinger-Moser inequalities of Adimurthi-Druet type in dimension two. *J. Differential Equations*, 258(9):3161–3193, 2015.